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Orthogonal polynomials on the unit circle via a polynomial mapping on the real line☆

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In memory of Professor Joaquin Bustoz, my teacher and my friend

Abstract

Let $\{\Phi_n\}_{n\geqslant 0}$ be a sequence of monic orthogonal polynomials on the unit circle (OPUC) with respect to a symmetric and finite positive Borel measure $d\mu$ on $[0,2\pi]$ and let $-1,\alpha_0,\alpha_1,\alpha_2,\ldots$ be the associated sequence of Verblunsky coefficients. In this paper we study the sequence $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ of monic OPUC whose sequence of Verblunsky coefficients is

$$-1,-b_1,-b_2,\ldots,-b_{N-1},\alpha_0,b_{N-1},\ldots,b_2,b_1,$$
$$\alpha_1,-b_1,-b_2,\ldots,-b_{N-1},\alpha_2,b_{N-1},\ldots,b_2,b_1,\alpha_3,\ldots$$

where $b_1,b_2,\ldots,b_{N-1}$ are $N-1$ fixed real numbers such that $b_j\in(-1,1)$ for all $j=1,2,\ldots,N-1$, so that $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ is also orthogonal with respect to a symmetric and finite positive Borel measure $d\widetilde{\mu}$ on the unit circle. We show that the sequences of monic orthogonal polynomials on the real line (OPRL) corresponding to $\{\Phi_n\}_{n\geqslant 0}$ and $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ (by Szegö's transformation) are related by some polynomial mapping, giving rise to a one-to-one correspondence between the monic OPUC $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ on the unit circle and a pair of monic OPRL on (a subset of) the interval $[-1,1]$. In particular we prove that

$$d\widetilde{\mu}(\theta)=\left|\zeta_{N-1}(\theta)\right|\left|\frac{\sin\theta}{\sin\vartheta_N(\theta)}\right|\frac{d\mu(\vartheta_N(\theta))}{\vartheta'_N(\theta)},$$

supported on (a subset of) the union of $2N$ intervals contained in $[0,2\pi]$ such that any two of these intervals have at most one common point, and where, up to an affine change in the variable, ζ_{N-1} and $\cos\vartheta_N$ are algebraic polynomials in $\cos\theta$ of degrees $N-1$ and N (respectively) defined only in terms of $\alpha_0,b_1,\ldots,b_{N-1}$. This measure induces a measure on the unit circle supported on the union of $2N$ arcs, pairwise symmetric with respect to the real axis. The restriction to symmetric measures (or real Verblunsky coefficients) is needed in order that Szegö's transformation may be applicable.
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1. Introduction

Throughout this paper we will use B. Simon's abbreviations OPUC and OPRL for orthogonal polynomials on the unit circle and orthogonal polynomials on the real line, resp. (cf. [32–34]). OPRL on several intervals have been a subject of investigation in the field of orthogonal polynomials (OPs) and special functions in the past 20 years. Often this kind of OPs can be generated via a polynomial mapping, starting with a given sequence of OPRL and then making a change of variable $x \mapsto T(x)$, T being a polynomial fulfilling certain properties. This way to generate OPs was considered in its general form in a pioneer paper by Bessis and Moussa [5], where several algebraic and analytical properties were discussed, including the orthogonality measure—which becomes associated with a contour integral on the Julia set related to the polynomial transformation. This construction was extended by Geronimo and Van Assche [13], having shown that if $\{p_n(\cdot;\mathrm{d}\sigma_0)\}_{n\geqslant 0}$ is a system of orthonormal polynomials with respect to some positive Borel measure $\mathrm{d}\sigma_0$, with support contained in the interval $[-1,1]$, and T is a polynomial of degree $N\geqslant 2$ with real and simple zeros, such that $|T(y_\nu)|\geqslant 1$ for all zeros $y_1,\ldots,y_{N-1}$ of T'—nowadays, such a polynomial T is called admissible (cf., e.g., [37]; for properties involving such polynomials, see [26])—, then there exists a positive Borel measure $\mathrm{d}\sigma$ and a sequence of polynomials $\{p_n(\cdot;\mathrm{d}\sigma)\}_{n\geqslant 0}$, orthonormal with respect to $\mathrm{d}\sigma$, such that the relation

$$p_{nN}(x;\mathrm{d}\sigma) = p_n(T(x);\mathrm{d}\sigma_0) \tag{1.1}$$

holds for all $n=0,1,2,\ldots$. If $\{p_n(\cdot;\mathrm{d}\sigma)\}_{n\geqslant 0}$ and $\{p_n(\cdot;\mathrm{d}\sigma_0)\}_{n\geqslant 0}$ are two sequences of OPs such that (1.1) holds, where T is an admissible polynomial, then we say that $\{p_n(\cdot;\mathrm{d}\sigma)\}_{n\geqslant 0}$ is obtained from $\{p_n(\cdot;\mathrm{d}\sigma_0)\}_{n\geqslant 0}$ by an admissible polynomial mapping. This kind of polynomial mappings arise in problems from Quantum Mechanics and Physics (see, e.g., [2,5,4,13] and the references therein) as well as in connection with the so-called sieved OPs (see, e.g., [1,6,8,13,17] and references therein). Also, similar transformation laws for OPs were used to solve some algebraic problems in matrix theory [20,10]. In this paper we explore this kind of transformations in another direction, more precisely to solve the following inverse problem (**P**) in below concerning OPUC. We recall that monic OPUC $\{\Phi_n\}_{n\geqslant 0}$ are characterized by the Szegö's recurrence relation

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \quad n=0,1,2,\ldots \quad (\Phi_0(z)=1) \tag{1.2}$$

with $|\Phi_n(0)|\neq 1$ for all $n=1,2,\ldots$, where $\Phi_n^*(z) := z^n\overline{\Phi_n(1/\overline{z})}$, the numbers

$$\alpha_n := -\overline{\Phi_{n+1}(0)}, \quad n=0,1,2,\ldots$$

being called the Verblunsky coefficients associated with $\{\Phi_n\}_{n\geqslant 0}$, after Simon [34]. In the literature the α_n's often are called Schur coefficients or reflection coefficients. The natural convention $\alpha_{-1} := -1$ is useful.

(**P**). *Let $\{\Phi_n\}_{n\geqslant 0}$ be a given sequence of monic OPUC, with real Verblunsky coefficients $-1,\alpha_0,\alpha_1,\ldots$ such that $\alpha_n\in(-1,1)$ for all $n=0,1,2,\ldots$, or, equivalently, orthogonal with respect to a symmetric measure on the unit circle. Let $\mathbf{b} := (b_1,b_2,\ldots,b_{N-1})$ be an $(N-1)$-tuple of real numbers with $b_j\in(-1,1)$ $(1\leqslant j\leqslant N-1)$ and introduce a monic OPUC $\{\widetilde{\Phi}_n\}_{n\geqslant 0}\equiv\{\widetilde{\Phi}_n(\cdot;\mathbf{b})\}_{n\geqslant 0}$ whose sequence of Verblunsky coefficients $-1,\widetilde{\alpha}_0,\widetilde{\alpha}_1,\ldots$ is defined by*

$$\widetilde{\alpha}_{nN-1} := \alpha_{n-1}, \quad \widetilde{\alpha}_{2nN+j-1} := -b_j, \quad \widetilde{\alpha}_{(2n+1)N+j-1} := b_{N-j}, \quad j=1,\ldots,N-1 \tag{1.3}$$

for all $n=0,1,2,\ldots$. The problem is to characterize all such sequences of OPUC $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$.

Since we are dealing with real Verblunsky coefficients, (1.3) can be rewritten as

$$\widetilde{\Phi}_{nN}(0) = \Phi_n(0), \quad \widetilde{\Phi}_{2nN+j}(0) = b_j, \quad \widetilde{\Phi}_{(2n+1)N+j}(0) = -b_{N-j}, \quad j=1,\ldots,N-1$$

for all $n=0,1,2,\ldots$.

We will give explicit expressions for the perturbed polynomials $\widetilde{\Phi}_n$'s in terms of the original polynomials Φ_n's, as well as explicit formulas for the Carathéodory function and the orthogonality measure on the unit circle for the perturbed polynomials.

The particular case when $\alpha_{2j}=$ constant for all $j\in\mathbb{N}_0$ is covered by the results of Peherstorfer and Steinbauer [30]. For $N=2$ problem (**P**) was partially analyzed in [21], where the orthogonality measure has been computed in such a case. The general case with complex Verblunsky coefficients $\alpha_0,\alpha_1,\alpha_2,\ldots$ and complex parameters $b_1,\ldots,b_{N-1}$ with moduli less than 1 in Problem (**P**), remains a very interesting open problem.

The restriction to real Verblunsky coefficients (or symmetric measures) is needed, since firstly we will transfer the problem from the unit circle into the real line, and then we solve it there by using the methods and results of Geronimo and Van Assche [13]. Then we mainly adapt the ideas from Peherstorfer and Steinbauer [30] (see also [35]), together with Szegö's transformation, to treat the unit circle case. This procedure is summarized in the following scheme

$$\begin{array}{ccc} (\Phi_n) & — & (\widetilde{\Phi}_n) \\ \downarrow & & \uparrow \\ (P_n) & \longrightarrow & (\widetilde{P}_n) \end{array} \tag{1.4}$$

where $\{P_n\}_{n\geqslant 0}$ and $\{\widetilde{P}_n\}_{n\geqslant 0}$ denote the sequences of monic OPRL corresponding to the sequences $\{\Phi_n\}_{n\geqslant 0}$ and $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$, respectively (by Szegö's transformation). As a key step in constructing the solution to problem (**P**) we will show that $\{\widetilde{P}_n\}_{n\geqslant 0}$ can be obtained from $\{P_n\}_{n\geqslant 0}$ by an admissible polynomial mapping so that (1.1) holds. Thus $\{\widetilde{P}_n\}_{n\geqslant 0}$ is orthogonal with respect to a measure supported on several intervals of the real line, and this fact enables us to show that $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ will be orthogonal on the unit circle with respect to a measure supported on several arcs of the unit circle. This kind of inverse problem on systems of OPUC with orthogonality measures supported on several arcs on the unit circle appear in several works (see, e.g., [28–30,12]).

From an algebraic point of view we start with a sequence of Verblunsky coefficients, say $-1, \alpha_0, \alpha_1, \alpha_2, \ldots,$ and then we make a perturbation of this sequence by inserting repeatedly blocks of $N-1$ real parameters in the following way:

$$-1, -b_1, \ldots, -b_{N-1}, \alpha_0, b_{N-1}, \ldots, b_1, \alpha_1, -b_1, \ldots, -b_{N-1}, \alpha_2, b_{N-1}, \ldots, b_1, \alpha_3, \ldots. \tag{1.5}$$

Thus, the main question is to describe, in terms of the original sequence $\{\Phi_n\}_{n\geqslant 0}$ and the vector $\mathbf{b}$, the sequence $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ of monic OPUC whose Verblunsky coefficients are given by (1.5).

When $b_1 = \cdots = b_{N-1} = 0$ then (1.5) yields the so-called sieved orthogonal polynomials on the unit circle, studied earlier by several authors, namely Marcellán and Sansigre [22,23] (who studied the algebraic properties of the perturbed sequence for the particular cases $N=2$ and 3), Badkov [3] (who firstly characterized the measure with respect to which the perturbed sequence is orthogonal), and by Ismail and Xin Li (who also gave for this special case the orthogonality measure for the perturbed sequence as well as the relation between the Carathéodory functions for the perturbed and the original OP sequences).

As a similar problem, one can consider a new perturbed OPUC characterized by a sequence of Verblunsky coefficients

$$-1, b_1, \ldots, b_{k-1}, \alpha_0, b_1, \ldots, b_{k-1}, \alpha_1, b_1, \ldots, b_{k-1}, \alpha_2, \ldots, \tag{1.6}$$

$b_1, \ldots, b_{k-1}$ being an $(k-1)$−tuple of complex numbers such that $|b_j| < 1$ for all $j = 1, \ldots, k-1$. Hence, denoting by $\{\Psi_n\}_{n\geqslant 0}$ the sequence of monic OPUC associated to (1.6)—assuming that for the original sequence $\{\Phi_n\}_{n\geqslant 0}$ the corresponding Verblunsky coefficients fulfill $|\alpha_n| < 1$ for all $n \geqslant 1$—, we have

$$\Psi_{kn}(0) = \Phi_n(0), \quad \Psi_{kn+j}(0) = -\overline{b_j}, \quad j = 1, \ldots, k-1$$

for all $n = 0, 1, 2, \ldots$. Thus, the main question associated with (1.6) is to describe the OPUC $\{\Psi_n\}_{n\geqslant 0}$, giving explicitly the orthogonality measure $\mathrm{d}\nu$ on the unit circle with respect to which it is orthogonal in terms of the orthogonality measure $\mathrm{d}\mu$ on the unit circle with respect to which $\{\Phi_n\}_{n\geqslant 0}$ is orthogonal. This question has been raised in [19]. When $\{\alpha_n\}_{n\geqslant 0}$ is a sequence of real numbers in the interval $(-1, 1)$—or, equivalently, $\mathrm{d}\mu$ is a symmetric measure on the unit circle—and $\{b_\ell\}_{\ell=1}^{k-1}$ is a symmetric sequence, in the sense that

$$b_{k-j} = -\overline{b_j} \quad (j = 1, 2, \ldots, k-1), \tag{1.7}$$

the problem has been completely solved by Peherstorfer and Steinbauer [30] (see also [35]). In fact, these authors shown that the relation between the Carathéodory functions associated with the measures $\mathrm{d}\mu$ and $\mathrm{d}\nu$ is given by

$$C(z; \mathrm{d}\nu) = \frac{W(z)}{\sqrt{R(z)}} C(\Theta(z); \mathrm{d}\mu), \tag{1.8}$$

where W and R are some polynomials and Θ a certain mapping (not necessarily a polynomial). It is clear that (1.8) characterizes $\mathrm{d}\nu$, by Stieltjes inversion formula (on the unit circle). The authors of [30] gave the explicit representation for $d\nu$ in terms of $d\mu$, and they shown that if $\{\theta_\ell\}_{\ell=1}^{2k}$ is the set of roots of a certain trigonometric equation $T(e^{i\theta})=L := 2\prod_{\ell=0}^{k-1}(1-|b_\ell|^2)$ and $\Gamma_\ell=\{e^{i\theta} : \theta\in[\theta_{2\ell-1},\theta_{2\ell}]\}$, then Θ is an analytic function on $\mathbb{C}\backslash\bigcup_{\ell=1}^{k}\Gamma_\ell$ which maps this domain onto the open unit disk

$$\mathbb{D} := \{z\in\mathbb{C}||z|<1\}.$$

Further, the restriction $\Theta|\Gamma_\ell : \Gamma_\ell\to\partial\mathbb{D} := \{z\in\mathbb{C}||z|=1\}$ is a continuous and bijective map and satisfies $\Theta(\Gamma_\ell)=\mathbb{D}$ and $\Theta(z)\in(-1,1)$ for $z\in\partial\mathbb{D}\backslash\bigcup_{\ell=1}^{k}\Gamma_\ell$.

If the b_j's in (1.6) are all real numbers belonging to the interval $(-1,1)$ and the symmetry condition (1.7) holds then there exists a connection between the OPRLs—corresponding to the OPUC's $\{\Phi_n\}_{n\geqslant 0}$ and $\{\Psi_n\}_{n\geqslant 0}$—by an admissible polynomial mapping (but we notice that this connection needs not to hold in general if (1.7) fails). This holds since if we consider (1.6) with $b_1,\ldots,b_{k-1}$ real and fulfilling (1.7), then we have $(b_1,\ldots,b_{k-1})=(-b_{k-1},\ldots,-b_1)$, so we see that for real Verblunsky coefficients (or symmetric measures) the perturbed OPUC described by (1.6) and satisfying (1.7) constitute a special case of the perturbed OPUC described by (1.5).

The structure of the paper is as follows. In Section 2 we recall some basic results concerning the general theory of OPs, both on the real line and on the unit circle. In Section 3 we will state the above mentioned connection by an admissible polynomial mapping between the OPRL corresponding to the OPUC. Then, using these connections, in Section 4 we give the answer to Problem (**P**), thus providing explicit representations of the new polynomials $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ in terms of the starting polynomials $\{\Phi_n\}_{n\geqslant 0}$, and determining both the Carathéodory function and the orthogonality measure for the new set of perturbed OPs $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$. We finish the paper giving the asymptotic behavior of the orthonormal polynomials $\{\widetilde{\varphi}_n\}_{n\geqslant 0}$ corresponding to $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ when the initial measure belongs to Szegö's class.

2. Background

For an updated overview on OPUC we refer to Simon [32–34]. Let $d\mu$ be a finite positive Borel measure on the interval $[0,2\pi]$ such that its support is an infinite set, and denote by $\{\Phi_n\}_{n\geqslant 0}$ the monic OPUC with respect to $d\mu$, so that

$$\int_0^{2\pi}\Phi_n(e^{i\theta})\overline{\Phi_m(e^{i\theta})}\,d\mu(\theta)=\lambda_n\delta_{nm}\quad(\lambda_n>0,\ n,m=0,1,2,\ldots).$$

It is well known that $\{\Phi_n\}_{n\geqslant 0}$ satisfies the Szegö's recurrence relation (1.2) with $|\Phi_n(0)|<1$ for all $n=1,2,\ldots$. Conversely, given a sequence of complex numbers $\{\alpha_n\}_{n\geqslant 0}$, with $|\alpha_n|<1$ for all $n=0,1,2,\ldots$, there exists a unique finite positive Borel measure $d\mu$ such that the corresponding monic OPUC, $\{\Phi_n\}_{n\geqslant 0}$, satisfies $\Phi_{n+1}(0)=-\overline{\alpha_n}$ for all $n=0,1,2,\ldots$. The Herglotz transform of a finite positive Borel measure $d\mu$ supported on $[0,2\pi]$ is defined by

$$C(z;d\mu) := \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta),\quad |z|<1. \tag{2.1}$$

In the open unit disk $C(z;d\mu)$ is an analytic function with positive real part, i.e., it is a Carathéodory function (C-function). Conversely, given a C-function, C, there exists a unique positive Borel measure $d\mu$ such that C is the corresponding Herglotz transform. This fact follows from the Stieltjes inversion formula,

$$\frac{1}{2}[\mu(\theta+0)+\mu(\theta-0)]=\text{const.}+\lim_{r\uparrow 1}\int_0^{\theta}\operatorname{Re}\{C(re^{i\phi})\}\,d\phi.$$

Thus, introducing the so called associated polynomials of second kind corresponding to $\{\Phi_n\}_{n\geqslant 0}$,

$$\Omega_n(z) := \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}[\Phi_n(e^{i\theta})-\Phi_n(z)]d\mu(\theta), \tag{2.2}$$

the following relations hold for $|z|<1$

$$\Phi_n(z)C(z;d\mu)+\Omega_n(z)=O(z^n),\qquad \Phi_n^*(z)C(z;d\mu)-\Omega_n^*(z)=O(z^{n+1})$$

for each $n=0,1,\ldots$ (cf. [27]).

OPUC are closely related to OPRL with respect to measures supported on a bounded closed interval of the real line. Let $\{P_n\}_{n\geqslant 0}$ be a monic OPRL with respect to some finite positive Borel measure $\mathrm{d}\sigma$ with (infinite) support contained in the interval $[-1,1]$. Then, writing $\sigma(\theta) := \mathrm{d}\sigma(\{t : 0\leqslant t<\theta\})$ and $\mu(\theta) := \mathrm{d}\mu(\{t : 0\leqslant t<\theta\})$, a positive Borel measure $\mathrm{d}\mu$ on the interval $[0,2\pi]$ can be defined by

$$\mu(\theta) := \begin{cases} \sigma(1)-\sigma(\cos\theta), & 0\leqslant\theta\leqslant\pi, \\ \sigma(1)+\sigma(\cos\theta), & \pi\leqslant\theta\leqslant 2\pi, \end{cases}$$

which one usually write as

$$\mathrm{d}\mu(\theta) \equiv |\mathrm{d}\sigma(\cos\theta)| \quad (0\leqslant\theta\leqslant 2\pi).$$

This is usually called the Szegö transformation (cf. [36,11,34]). Associated with this measure $\mathrm{d}\mu$ there exists a monic OPUC, say $\{\Phi_n\}_{n\geqslant 0}$. Then, the coefficients of each polynomial Φ_n are real and the relation between $\{\Phi_n\}_{n\geqslant 0}$ and $\{P_n\}_{n\geqslant 0}$ can be deduced from

$$\begin{bmatrix} P_n(x) \\ (z-z^{-1})Q_{n-1}(x) \end{bmatrix} = \begin{bmatrix} \varepsilon_n & \varepsilon_n \\ \delta_n & -\delta_n \end{bmatrix} \begin{bmatrix} z^{-n}\Phi_{2n}(z) \\ z^n\Phi_{2n}(z^{-1}) \end{bmatrix}, \quad x=\frac{z+z^{-1}}{2}, \tag{2.3}$$

where

$$\varepsilon_n := \frac{2^{-n}}{1+\Phi_{2n}(0)}, \quad \delta_n := \frac{2^{1-n}}{1-\Phi_{2n}(0)} \tag{2.4}$$

and $\{Q_n\}_{n\geqslant 0}$ is the monic OPRL with respect to the measure $(1-x^2)\,\mathrm{d}\sigma$ (see [36, p. 294]). The interest of introducing the sequence $\{Q_n\}_{n\geqslant 0}$ is that from the above matrix relation one can give explicitly the Φ_n's in terms of the P_n's and the Q_n's. Moreover, for the coefficients $\{\beta_n,\gamma_{n+1}\}_{n\geqslant 0}$ of the three-term recurrence relation satisfied by the monic OPRL $\{P_n\}_{n\geqslant 0}$,

$$xP_n(x) = P_{n+1}(x)+\beta_n P_n(x)+\gamma_n P_{n-1}(x), \quad n=1,2,\ldots \tag{2.5}$$

$(P_0(x)\equiv 1,\ P_1(x)=x-\beta_0;\ \gamma_n>0$ for all $n=1,2,\ldots)$, the relations

$$\begin{aligned} 2\beta_n &= \Phi_{2n-1}(0)[1-\Phi_{2n}(0)]-\Phi_{2n+1}(0)[1+\Phi_{2n}(0)], \\ 4\gamma_{n+1} &= [1-\Phi_{2n+2}(0)][1-\Phi_{2n+1}^2(0)][1+\Phi_{2n}(0)] \end{aligned} \tag{2.6}$$

hold for every $n=0,1,\ldots$ ([14, p. 67]). Using these formulas it can be shown that

$$Q_{n-1}(x) = \frac{1}{x^2-1}\{P_{n+1}(x)-a_n^{(1)}P_n(x)-a_n^{(0)}P_{n-1}(x)\}, \tag{2.7}$$

for all $n=1,2,\ldots$, where

$$a_n^{(1)} := -\beta_n+\Phi_{2n-1}(0), \quad a_n^{(0)} := \frac{1+\Phi_{2n}(0)}{1-\Phi_{2n}(0)}\gamma_n. \tag{2.8}$$

Notice that formula (2.7) is the classical Christoffel formula when the quadratic symmetric polynomial perturbation of a real measure is introduced (see, e.g., [9, Chapter 1]). Finally, recall that the Stieltjes function of the positive finite Borel measure $\mathrm{d}\sigma$, supported on a subset of $[-1,1]$, is defined and characterized by

$$F(x;\mathrm{d}\sigma) := \int_{-\infty}^{+\infty}\frac{\mathrm{d}\sigma(t)}{t-x} = -u_0\lim_{n\to+\infty}\frac{P_{n-1}^{(1)}(x)}{P_n(x)}, \quad x\in\mathbb{C}\backslash[-1,1]$$

where $u_0 := \int_{-\infty}^{+\infty}\mathrm{d}\sigma(t)$ and $P_{n-1}^{(1)}$ is the monic associated polynomial of the first kind of degree $n-1$ corresponding to the sequence $\{P_n\}_{n\geqslant 0}$. Then the Stieltjes and the Carathéodory functions associated to $\mathrm{d}\sigma$ and $\mathrm{d}\mu$ are related by

$$F(x;\mathrm{d}\sigma) = \frac{2z}{1-z^2}C(z;\mathrm{d}\mu), \quad x=\frac{z+z^{-1}}{2} \quad (|x|\to+\infty) \tag{2.9}$$

(cf. [14, p. 64]).

3. Connection with a polynomial mapping on the real line

Since we impose in Problem (**P**) that the starting measure $\mathrm{d}\mu$ with respect to which $\{\Phi_n\}_{n\geqslant 0}$ is orthogonal is symmetric, or, equivalently, the Verblunsky coefficients α_n are real numbers belonging to $(-1,1)$, and that $b_1,\ldots,b_{N-1}$ also belong to $(-1,1)$, then Szegö's transformation between OPs on the real line and on the unit circle is applicable. Let $\{P_n\}_{n\geqslant 0}$ and $\{\widetilde{P}_n\}_{n\geqslant 0}$ be the sequences of monic OPRL corresponding to $\{\Phi_n\}_{n\geqslant 0}$ and $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ (resp.).

In this section we will show that $\{P_n\}_{n\geqslant 0}$ and $\{\tilde{P}_n\}_{n\geqslant 0}$ are connected by an admissible polynomial mapping in the sense described in the introduction, i.e., there exists an admissible polynomial T, of degree N, such that (up to an affine change in the variable)

$$\widetilde{P}_{nN}(x)=P_n(T(x)),\quad n=0,1,2,\ldots\ .$$

This connection enables us to study, on the real line, the properties of $\{\widetilde{P}_n\}_{n\geqslant 0}$ from those of $\{P_n\}_{n\geqslant 0}$. Using these properties, in the next section we will "return to the unit circle" and determine the properties of $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ in terms of the properties of $\{\Phi_n\}_{n\geqslant 0}$, according to (1.4), thus giving the solution for Problem (**P**).

3.1. Preliminary results

Let $\{\beta_n\}_{n\geqslant 0}$ and $\{\gamma_n\}_{n\geqslant 1}$, with β_n real and $\gamma_{n+1}>0$ for all $n=0,1,2,\ldots$, be the sequences of coefficients which appear in the three-term recurrence relation for $\{P_n\}_{n\geqslant 0}$,

$$xP_n(x)=P_{n+1}(x)+\beta_nP_n(x)+\gamma_nP_{n-1}(x),\quad n=0,1,2,\ldots \tag{3.1}$$

with initial conditions $P_{-1}(x)\equiv 0$ and $P_0(x)\equiv 1$.

In the sequel, we will assume $N\geqslant 3$. The case $N=2$ has been partially treated in [21] and it can be directly worked out with the ideas presented here. Put

$$d:=\left(\prod_{j=1}^{N-1}\frac{1-b_j^2}{2}\right)^{1/N},\quad \beta:=-\frac{b_1}{d},\quad \gamma:=\frac{(1-b_2)(1-b_1^2)}{2d^2}. \tag{3.2}$$

In the sequel we will introduce some parameters $\widehat{\beta}_n\in\mathbb{R}$ and $\widehat{\gamma}_n>0$. Our goal is to show that the associated sequence of monic OPRL, denoted by $\{\widehat{P}_n\}_{n\geqslant 0}$, satisfies $d^n\widehat{P}_n(x/d)=\widetilde{P}_n(x)$ for all $n=1,2,\ldots$, where $\{\widetilde{P}_n\}_{n\geqslant 0}$ is the monic OPRL corresponding to the OPUC $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$.

Since, together with the b_j's, our data are the Verblunsky coefficients of $\{\Phi_n(z)\}_{n\geqslant 0}$, we can introduce two sequences $\{\widehat{\beta}_n\}_{n\geqslant 0}$ and $\{\widehat{\gamma}_n\}_{n\geqslant 1}$ of real numbers by

$$2d\widehat{\beta}_{nN+j}:=\begin{cases}-2b_1, & j=0,\\ b_{2j-1}(1-b_{2j}) & \\ \quad -b_{2j+1}(1+b_{2j}), & j=1,2,\ldots,\left\lfloor\dfrac{N}{2}\right\rfloor-1,\\ b_{N-2}(1-b_{N-1}) & \\ \quad -(1+b_{N-1})\Phi_{2n+1}(0), & j=\dfrac{N-1}{2}\text{ and }N\text{ is odd},\\ 2b_{N-1}, & j=\dfrac{N}{2}\text{ and }N\text{ is even}\\ b_{N-2}(1-b_{N-1}) & \\ \quad +(1+b_{N-1})\Phi_{2n+1}(0), & j=\dfrac{N+1}{2}\text{ and }N\text{ is odd},\\ b_{2N-2j-1}(1-b_{2N-2j}) & \\ \quad -b_{2N-2j+1}(1+b_{2N-2j}), & j=\left\lfloor\dfrac{N+1}{2}\right\rfloor+1,\ldots,N-1\end{cases} \tag{3.3}$$

and

$$4d^2\widehat{\gamma}_{nN+j} := \begin{cases} (1-b_2)(1-b_1^2)[1-\Phi_{2n}(0)], & j=0, \\ (1-b_2)(1-b_1^2)[1+\Phi_{2n}(0)], & j=1, \\ (1-b_{2j})(1-b_{2j-1}^2)(1+b_{2j-2}), & j=2,\ldots,\left\lfloor \dfrac{N-1}{2} \right\rfloor, \\ (1+b_{N-2})(1-b_{N-1}^2) & \\ \quad\times[1-\Phi_{2n+1}(0)], & j=\dfrac{N}{2} \text{ and } N \text{ is even}, \\ (1+b_{N-1})^2[1-\Phi_{2n+1}^2(0)], & j=\dfrac{N+1}{2} \text{ and } N \text{ is odd}, \\ (1+b_{N-2})(1-b_{N-1}^2) & \\ \quad\times[1+\Phi_{2n+1}(0)], & j=\dfrac{N}{2}+1 \text{ and } N \text{ is even}, \\ (1+b_{2N-2j})(1-b_{2N-2j+1}^2) & \\ \quad\times(1-b_{2N-2j+2}), & j=\left\lfloor \dfrac{N}{2} \right\rfloor+2,\ldots,N-1 \end{cases} \tag{3.4}$$

for all $n=0,1,2,\ldots$.

We point out the following properties, which can be verified by a straightforward computation

$$\widehat{\beta}_{nN} = \beta, \quad n=0,1,2,\ldots, \tag{3.5}$$

$$\widehat{\beta}_{nN+j} = \widehat{\beta}_j = \widehat{\beta}_{N-j}, \quad j=1,2,\ldots,\left\lfloor \frac{N}{2} \right\rfloor - 1, \quad n=0,1,2,\ldots, \tag{3.6}$$

$$\widehat{\gamma}_{nN+j+1} = \widehat{\gamma}_{j+1} = \widehat{\gamma}_{N-j}, \quad j=1,2,\ldots,\left\lfloor \frac{N-1}{2} \right\rfloor - 1, \quad n=0,1,2,\ldots, \tag{3.7}$$

$$\prod_{j=0}^{N-1} \widehat{\gamma}_{nN-j} = \gamma_n, \quad n=1,2,\ldots, \tag{3.8}$$

$$\widehat{\gamma}_{nN} + \widehat{\gamma}_{nN+1} = \gamma, \quad n=0,1,2,\ldots \tag{3.9}$$

(with $\widehat{\gamma}_0 \equiv 0$). Notice that (3.8) comes from the Geronimus relation (2.6), and it motivates the choice of d as in (3.2). Also, we see that if N is even then $\widehat{\beta}_{nN+j}$ is also independent of n for each $j=0,1,\ldots,N-1$.

For $r, s \in \mathbb{N}_0$, we define $\Delta_{r,s}(x;n)$, with $x \in \mathbb{C}$ and $n \in \mathbb{N}_0$, as

$$\Delta_{r,s}(x;n) := \begin{cases} 0 & \text{if } r \geqslant s+2, \\ 1 & \text{if } r = s+1, \\ \det(\text{tridiag } [x-\widehat{\beta}_{nN+i}, \widehat{\gamma}_{nN+i}, 1]_{i=r}^{s}) & \text{if } r \leqslant s, \end{cases}$$

where we have used the notation $\mathrm{tridiag}[e_i, f_i, g_i]_{i=r}^{s}$ $(0 \leqslant r \leqslant s)$ to denote the tridiagonal matrix of order $s - r + 1$ defined by

$$\mathrm{tridiag}[e_i, f_i, g_i]_{i=r}^{s} := \begin{bmatrix} e_r & g_r & 0 & \dots & 0 & 0 & 0 \\ f_{r+1} & e_{r+1} & g_{r+1} & \dots & 0 & 0 & 0 \\ 0 & f_{r+2} & e_{r+2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & e_{s-2} & g_{s-2} & 0 \\ 0 & 0 & 0 & \dots & f_{s-1} & e_{s-1} & g_{s-1} \\ 0 & 0 & 0 & \dots & 0 & f_s & e_s \end{bmatrix}.$$

These determinants $\Delta_{r,s}(x;n)$ were introduced by Charris and Ismail in [7] for the symmetric case and by Charris, Ismail, and Monsalve in [7,8] for the general case. Notice that $\Delta_{r,s}(x;n)$ is a polynomial in x of degree $s - r + 1$ (with the usual convention that polynomials with negative degree are zero) which satisfies

$$\Delta_{r,s}(x;n) = \Delta_{r,r+k}(x;n)\Delta_{r+k+1,s}(x;n) - \widehat{\gamma}_{nN+r+k+1}\Delta_{r,r+k-1}(x;n)\Delta_{r+k+2,s}(x;n) \tag{3.10}$$

for all $r, s, k, n = 0, 1, \dots$. This property can be easily checked (otherwise, see Muir [24, p. 518]). In particular, if we choose $k = 0$ we find

$$\Delta_{r,s}(x;n) = (x - \widehat{\beta}_{nN+r})\Delta_{r+1,s}(x;n) - \widehat{\gamma}_{nN+r+1}\Delta_{r+2,s}(x;n) \tag{3.11}$$

for all $r, s, n = 0, 1, \dots$, and for $k = s - r - 1$ we get

$$\Delta_{r,s}(x;n) = (x - \widehat{\beta}_{nN+s})\Delta_{r,s-1}(x;n) - \widehat{\gamma}_{nN+s}\Delta_{r,s-2}(x;n) \tag{3.12}$$

for $r, s, n = 0, 1, \dots$.

It is clear that $\widehat{\beta}_n$ is real and $\widehat{\gamma}_{n+1} > 0$ for all $n = 0, 1, 2, \dots$. Thus, according to the spectral theorem for OPs (see Ismail [18, pp. 31–32]), the sequence of polynomials $\{\widehat{P}_n\}_{n\geqslant 0}$ characterized by the three-term recurrence relation

$$x\widehat{P}_n(x) = \widehat{P}_{n+1}(x) + \widehat{\beta}_n\widehat{P}_n(x) + \widehat{\gamma}_n\widehat{P}_{n-1}(x), \quad n = 0, 1, 2, \dots , \tag{3.13}$$

with initial conditions $\widehat{P}_{-1}(x) \equiv 0$ and $\widehat{P}_0(x) \equiv 1$, is a monic OPRL with respect to some positive measure.

It follows from (3.11) and the standard theory of OPs that, in terms of the monic numerator polynomials corresponding to the sequence $\{\widehat{P}_n\}_{n\geqslant 0}$, we have the representation

$$\Delta_{r,s}(x;n) = \widehat{P}_{s-r+1}^{(nN+r)}(x), \quad r, s, n = 0, 1, 2, \dots . \tag{3.14}$$

According to the block structure of the definition of the sequences $\{\widehat{\beta}_n\}_{n\geqslant 0}$ and $\{\widehat{\gamma}_n\}_{n\geqslant 1}$, it is convenient to write the above recurrence in the following way:

$$(x - \widehat{\beta}_{nN+j})\widehat{P}_{nN+j}(x) = \widehat{P}_{nN+j+1}(x) + \widehat{\gamma}_{nN+j}\widehat{P}_{nN+j-1}(x),$$
$$j = 0, 1, \dots, N-1, \quad n = 0, 1, 2, \dots, \tag{3.15}$$

with initial conditions $\widehat{P}_{-1}(x) \equiv 0$ and $\widehat{P}_0(x) \equiv 1$. We now show that this sequence $\{\widehat{P}_n\}_{n\geqslant 0}$ can be obtained from $\{P_n\}_{n\geqslant 0}$ by an admissible polynomial mapping. For that, notice first that, for each n, (3.15) is a system of N equations of the type (1.1) studied in [8]. Therefore, following section 2 of [8], and observing that, in the notation of [8] we have $\Delta_{i,j}(x;n) \equiv \Delta_n(i+1, j)$, it follows from (2.5) and (2.6) in [8] that

$$\Delta_{1,N-1}(x;n)\widehat{P}_{nN+j}(x) = \Delta_{1,j-1}(x;n)\widehat{P}_{(n+1)N}(x) + \left(\prod_{i=1}^{j}\widehat{\gamma}_{nN+i}\right)\Delta_{j+1,N-1}(x;n)\widehat{P}_{nN}(x) \tag{3.16}$$

for all $j=0,1,\ldots,N-1$ and $n=0,1,2,\ldots$, and

$$\widehat{\gamma}_{nN}\Delta_{1,N-1}(x;n)\widehat{P}_{nN-1}(x)=-\widehat{P}_{(n+1)N}(x)+[(x-\widehat{\beta}_{nN})\Delta_{1,N-1}(x;n)-\widehat{\gamma}_{nN+1}\Delta_{2,N-1}(x;n)]\widehat{P}_{nN}(x) \tag{3.17}$$

for all $n=0,1,2,\ldots$. In (3.16) replace n by $n-1$ and then take $j=N-1$ to find

$$\Delta_{1,N-1}(x;n-1)\widehat{P}_{nN-1}(x)=\Delta_{1,N-2}(x;n-1)\widehat{P}_{nN}(x)+\left(\prod_{i=1}^{N-1}\widehat{\gamma}_{(n-1)N+i}\right)\widehat{P}_{(n-1)N}(x),$$

which, after multiplication by $\widehat{\gamma}_{nN}$ and taking into account that

$$\widehat{\gamma}_{nN}\prod_{i=1}^{N-1}\widehat{\gamma}_{(n-1)N+i}=\prod_{i=1}^{N}\widehat{\gamma}_{(n-1)N+i}=\prod_{i=0}^{N-1}\widehat{\gamma}_{nN-i}=\gamma_n$$

(the last identity follows from (3.8)), gives

$$\widehat{\gamma}_{nN}\Delta_{1,N-1}(x;n-1)\widehat{P}_{nN-1}(x)=\widehat{\gamma}_{nN}\Delta_{1,N-2}(x;n-1)\widehat{P}_{nN}(x)+\gamma_n\widehat{P}_{(n-1)N}(x) \tag{3.18}$$

for all $n=0,1,2,\ldots$. Now, multiply both sides of (3.17) by $\Delta_{1,N-1}(x;n-1)$ and both sides of (3.18) by $\Delta_{1,N-1}(x;n)$, and then take the difference of the resulting equalities to obtain

$$\begin{aligned}&\Delta_{1,N-1}(x;n-1)\widehat{P}_{(n+1)N}(x)+\gamma_n\Delta_{1,N-1}(x;n)\widehat{P}_{(n-1)N}(x)\\&\quad=\{\Delta_{1,N-1}(x;n-1)[(x-\widehat{\beta}_{nN})\Delta_{1,N-1}(x;n)-\widehat{\gamma}_{nN+1}\Delta_{2,N-1}(x;n)]\\&\qquad-\widehat{\gamma}_{nN}\Delta_{1,N-1}(x;n)\Delta_{1,N-2}(x;n-1)\}\widehat{P}_{nN}(x),\quad n=0,1,2,\ldots.\end{aligned} \tag{3.19}$$

In order to proceed, we need the following:

Lemma 3.1. *For every $n=0,1,2,\ldots$ the relations*

$$\Delta_{i,N-i}(x;n)=\Delta_{i,N-i}(x;0),\quad 1\leqslant i\leqslant\lfloor N/2\rfloor, \tag{3.20}$$

$$\beta_n-\widehat{\gamma}_{nN+1}\Delta_{2,N-1}(x;n)-\widehat{\gamma}_{nN}\Delta_{1,N-2}(x;n-1)=\beta_0-\gamma\Delta_{2,N-1}(x;0), \tag{3.21}$$

$$\Phi_{2n+1}(0)-\gamma\Delta_{1,N-2}(x;n)=\beta_0-\gamma\Delta_{2,N-1}(x;0) \tag{3.22}$$

hold, so that the left-hand sides of (3.20)–(3.22) *are independent of n.*

Proof. The proof of (3.22) follows by using the ideas of the proof of (3.21), so we only prove (3.20) and (3.21). We need to divide the proof in two cases, according to the parity of N. We will assume that N is even (the case when N is odd can be handled similarly). It is clear that (3.20) holds for $i=N/2$, since

$$\Delta_{N/2,N/2}(x;n)=x-\widehat{\beta}_{nN+N/2}=x-\widehat{\beta}_{N/2}\quad\text{independent of } n.$$

In order to prove (3.20) for a fixed $1\leqslant i\leqslant N/2-1$, notice that if N is even then the only elements which depend on n in the determinant $\Delta_{i,N-i}(x;n)$ are $\widehat{\gamma}_{nN+N/2}$ and $\widehat{\gamma}_{nN+N/2+1}$. This suggests us to apply (3.10) with $r=i$, $s=N-i$ and $k=N/2-i$, so that

$$\Delta_{i,N-i}(x;n)=\Delta_{i,N/2}(x;n)\Delta_{N/2+1,N-i}(x;n)-\widehat{\gamma}_{nN+N/2+1}\Delta_{i,N/2-1}(x;n)\Delta_{N/2+2,N-i}(x;n) \tag{3.23}$$

for all $n=0,1,2,\ldots$. Now, by (3.12), we have

$$\Delta_{i,N/2}(x;n)=(x-\widehat{\beta}_{N/2})\Delta_{i,N/2-1}(x;n)-\widehat{\gamma}_{nN+N/2}\Delta_{i,N/2-2}(x;n) \tag{3.24}$$

for all $n = 0, 1, 2, \ldots$. Furthermore, for every $n = 0, 1, 2, \ldots$, we can write

$$\begin{aligned}
\Delta_{N/2+1,N-i}(x;n) &= \det(\text{tridiag}\ [x-\widehat{\beta}_{nN+j}, \widehat{\gamma}_{nN+j}, 1]_{j=N/2+1}^{N-i}) \\
&= \det(\text{tridiag}\ [x-\widehat{\beta}_{nN+j}, \widehat{\gamma}_{nN+j}^{1/2}, \widehat{\gamma}_{nN+j}^{1/2}]_{j=N/2+1}^{N-i}) \\
&= \det(\text{tridiag}\ [x-\widehat{\beta}_{nN+j}, \widehat{\gamma}_{nN+j}^{1/2}, \widehat{\gamma}_{nN+j}^{1/2}]_{j=i}^{N/2-1}) \\
&= \Delta_{i,N/2-1}(x;n) \\
&= \Delta_{i,N/2-1}(x;0),
\end{aligned} \tag{3.25}$$

where the third equality can be justified by (3.6)–(3.7) and an elementary property of determinants (according to which if two determinants of the same order m are such that the first row of one of them is, when reversed, the last row of the other, the second row, when reversed, the $(m-1)$st row of the other, and so on, then the two determinants are equal), and the last equality is also justified by (3.6)–(3.7). In the same way we can show that

$$\Delta_{N/2+2,N-i}(x;n) = \Delta_{i,N/2-2}(x;n) = \Delta_{i,N/2-2}(x;0), \quad n = 0, 1, 2, \ldots . \tag{3.26}$$

Therefore, by using (3.24), (3.25) and (3.26) in (3.23), and observing that

$$\widehat{\gamma}_{nN+N/2} + \widehat{\gamma}_{nN+N/2+1} = (1+b_{N-2})(1-b_{N-1}^2)/2d^2 \quad \text{independent of } n,$$

we obtain

$$\Delta_{i,N-i}(x;n) = (x-\widehat{\beta}_{N/2})\{\Delta_{i,N/2-1}(x;0)\}^2 - (\widehat{\gamma}_{N/2}+\widehat{\gamma}_{N/2+1})\Delta_{i,N/2-1}(x;0)\Delta_{i,N/2-2}(x;0)$$

for all $n = 0, 1, 2, \ldots$, from which (3.20) follows.

To prove (3.21), we start by applying (3.12) to $\Delta_{2,N-1}$ and (3.11) to $\Delta_{1,N-2}$, in order to write

$$\begin{aligned}
\Delta_{2,N-1}(x;n) &= (x-\widehat{\beta}_{N-1})\Delta_{2,N-2}(x;n) - \widehat{\gamma}_{N-1}\Delta_{2,N-3}(x;n), \\
\Delta_{1,N-2}(x;n-1) &= (x-\widehat{\beta}_1)\Delta_{2,N-2}(x;n-1) - \widehat{\gamma}_2\Delta_{3,N-2}(x;n-1),
\end{aligned}$$

or, according to (3.20) and (3.6)–(3.7),

$$\begin{aligned}
\Delta_{2,N-1}(x;n) &= (x-\widehat{\beta}_1)\Delta_{2,N-2}(x;0) - \widehat{\gamma}_2\Delta_{2,N-3}(x;n), \\
\Delta_{1,N-2}(x;n-1) &= (x-\widehat{\beta}_1)\Delta_{2,N-2}(x;0) - \widehat{\gamma}_2\Delta_{3,N-2}(x;n-1),
\end{aligned}$$

hence

$$\begin{aligned}
\widehat{\gamma}_{nN+1}\Delta_{2,N-1}(x;n) + \widehat{\gamma}_{nN}\Delta_{1,N-2}(x;n-1) = \widehat{\gamma}_1(x-\widehat{\beta}_1)\Delta_{2,N-2}(x;0) - \widehat{\gamma}_2\{&\widehat{\gamma}_{nN+1}\Delta_{2,N-3}(x;n) \\
&+ \widehat{\gamma}_{nN}\Delta_{3,N-2}(x;n-1)\}
\end{aligned} \tag{3.27}$$

(we also have used (3.9)). Now, we apply (3.11) to $\Delta_{2,N-3}$ and (3.12) to $\Delta_{3,N-2}$, and proceed as before to obtain

$$\begin{aligned}
\Delta_{2,N-3}(x;n) &= (x-\widehat{\beta}_2)\Delta_{3,N-3}(x;n) - \widehat{\gamma}_3\Delta_{4,N-3}(x;n) \\
&= (x-\widehat{\beta}_2)\Delta_{3,N-3}(x;0) - \widehat{\gamma}_3\Delta_{4,N-3}(x;n), \\
\Delta_{3,N-2}(x;n-1) &= (x-\widehat{\beta}_{N-2})\Delta_{3,N-3}(x;n-1) - \widehat{\gamma}_{N-2}\Delta_{3,N-4}(x;n-1) \\
&= (x-\widehat{\beta}_2)\Delta_{3,N-3}(x;0) - \widehat{\gamma}_3\Delta_{3,N-4}(x;n-1)
\end{aligned}$$

and then, by using (3.27) as well as (3.9), one sees that the left-hand side of (3.27) is

$$\begin{aligned}
&\widehat{\gamma}_1(x-\widehat{\beta}_1)\Delta_{2,N-2}(x;0) - \widehat{\gamma}_1\widehat{\gamma}_2(x-\widehat{\beta}_2)\Delta_{3,N-3}(x;0) \\
&\quad + \widehat{\gamma}_2\widehat{\gamma}_3\{\widehat{\gamma}_{nN+1}\Delta_{4,N-3}(x;n) + \widehat{\gamma}_{nN}\Delta_{3,N-4}(x;n-1)\}.
\end{aligned}$$

Now, we apply (3.12) to $\Delta_{4,N-3}$ and (3.11) to $\Delta_{3,N-4}$, and proceed as before, and then if we apply successively the same procedure, at the end we will get at the expression

$$\begin{aligned}\widehat{\gamma}_{nN+1}\Delta_{2,N-1}(x;n)+\widehat{\gamma}_{nN}\Delta_{1,N-2}(x;n-1) &= \sum_{i=2}^{N/2-1}(-1)^i\widehat{\gamma}_1\widehat{\gamma}_2\ldots\widehat{\gamma}_{i-1}(x-\widehat{\beta}_{i-1})\Delta_{i,N-i}(x;0)\\ &\quad+(-1)^{N/2}\widehat{\gamma}_2\widehat{\gamma}_3\ldots\widehat{\gamma}_{N/2-1}E_N(x;n),\end{aligned}\tag{3.28}$$

where

$$E_N(x;n):=\begin{cases}\widehat{\gamma}_{nN+1}\Delta_{N/2,N/2+1}(x;n)+\widehat{\gamma}_{nN}\Delta_{N/2-1,N/2}(x;n-1) & \text{if } N/2 \text{ is even}\\ \widehat{\gamma}_{nN+1}\Delta_{N/2-1,N/2}(x;n)+\widehat{\gamma}_{nN}\Delta_{N/2,N/2+1}(x;n-1) & \text{if } N/2 \text{ is odd.}\end{cases}\tag{3.29}$$

Now, from the definition of the $\widehat{\gamma}_i$'s we find

$$\widehat{\gamma}_2\widehat{\gamma}_3\ldots\widehat{\gamma}_{N/2-1}=\frac{(4d^2)^2}{2(1+b_{N-2})(1-b_{N-1}^2)(1-b_2)(1-b_1^2)}.$$

Hence, taking into account that $\widehat{\beta}_{N/2+1}=\widehat{\beta}_{N/2-1}$, as well as (3.9) and the definition of the $\Delta_{r,s}$'s, we can write

$$\begin{aligned}&\widehat{\gamma}_{nN+1}\Delta_{N/2,N/2+1}(x;n)+\widehat{\gamma}_{nN}\Delta_{N/2-1,N/2}(x;n-1)\\ &\quad=\gamma(x-\widehat{\beta}_{N/2-1})(x-\widehat{\beta}_{N/2})-\{\widehat{\gamma}_{nN+1}\widehat{\gamma}_{nN+N/2+1}+\widehat{\gamma}_{nN}\widehat{\gamma}_{(n-1)N+N/2}\}\\ &\quad=\gamma(x-\widehat{\beta}_{N/2-1})(x-\widehat{\beta}_{N/2})\\ &\qquad-\frac{2(1+b_{N-2})(1-b_{N-1}^2)(1-b_2)(1-b_1^2)}{(4d^2)^2}(1-\beta_n)\\ &\quad=\widehat{\gamma}_1(x-\widehat{\beta}_{N/2-1})(x-\widehat{\beta}_{N/2})-(\widehat{\gamma}_2\widehat{\gamma}_3\ldots\widehat{\gamma}_{N/2-1})^{-1}(1-\beta_n),\end{aligned}$$

where, for the second equality, we have used (2.6). Similarly, one proves

$$\begin{aligned}&\widehat{\gamma}_{nN+1}\Delta_{N/2-1,N/2}(x;n)+\widehat{\gamma}_{nN}\Delta_{N/2,N/2+1}(x;n-1)\\ &\quad=\widehat{\gamma}_1(x-\widehat{\beta}_{N/2-1})(x-\widehat{\beta}_{N/2})-(\widehat{\gamma}_2\widehat{\gamma}_3\ldots\widehat{\gamma}_{N/2-1})^{-1}(1+\beta_n).\end{aligned}$$

Therefore, we conclude that (3.29) becomes

$$E_N(x;n)=\widehat{\gamma}_1(x-\widehat{\beta}_{N/2-1})(x-\widehat{\beta}_{N/2})-(\widehat{\gamma}_2\widehat{\gamma}_3\ldots\widehat{\gamma}_{N/2-1})^{-1}[1-(-1)^{N/2}\beta_n],$$

from which, by (3.28) and taking into account that $x-\widehat{\beta}_{N/2}=\Delta_{N/2,N-N/2}(x;0)$, we obtain

$$\begin{aligned}&\beta_n-\widehat{\gamma}_{nN+1}\Delta_{2,N-1}(x;n)-\widehat{\gamma}_{nN}\Delta_{1,N-2}(x;n-1)\\ &\quad=(-1)^{N/2}-\sum_{i=2}^{N/2}(-1)^i\widehat{\gamma}_1\widehat{\gamma}_2\ldots\widehat{\gamma}_{i-1}(x-\widehat{\beta}_{i-1})\Delta_{i,N-i}(x;0),\end{aligned}$$

which proves (3.21) when N is even. □

Remark 3.2. The motivation for the previous Lemma comes from the known fact that (3.20) for $i=1$ as well as (3.21) are necessary conditions in order that $\{\widehat{P}_n\}_{n\geqslant 0}$ be obtained from $\{P_n\}_{\geqslant 0}$ by an admissible polynomial mapping (cf. [5], [13, Theorem 6], [8, Theorem 4.1 and Remark 4.2]).

3.2. Description of the polynomial mapping in $\mathbb{R}$

We now introduce the polynomial of degree N defined by

$$\pi_N(x) := \Delta_{0,N-1}(x;0) - \Phi_1(0). \tag{3.30}$$

From (3.11) and (3.21) we get

$$\pi_N(x) = (x-\beta)\Delta_{1,N-1}(x;0) + \beta_n - \widehat{\gamma}_{nN+1}\Delta_{2,N-1}(x;n) - \widehat{\gamma}_{nN}\Delta_{1,N-2}(x;n-1). \tag{3.31}$$

Then use (3.19), (3.20) and (3.31) to obtain

$$\widehat{P}_{(n+1)N}(x) = (\pi_N(x) - \beta_n)\widehat{P}_{nN}(x) - \gamma_n \widehat{P}_{(n-1)N}(x), \quad n = 0, 1, 2, \ldots,$$

from which one easily proves by induction that

$$\widehat{P}_{nN}(x) = P_n(\pi_N(x)), \quad n = 0, 1, 2, \ldots. \tag{3.32}$$

We are ready to show how the sequences $\{P_n\}_{n\geqslant 0}$ and $\{\widetilde{P}_n\}_{n\geqslant 0}$ are related.

Theorem 3.3. *Denote by $\{P_n\}_{n\geqslant 0}$ and $\{\widetilde{P}_n\}_{n\geqslant 0}$ the sequences of monic OPRL corresponding to the sequences of OPUC $\{\Phi_n\}_{n\geqslant 0}$ and $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ introduced in* Problem (**P**). *Let π_N be the monic polynomial of degree N defined by*

$$\pi_N(x) := \Delta_{0,N-1}(x;0) - \Phi_1(0).$$

Then, the relation between $\{P_n\}_{n\geqslant 0}$ and $\{\widetilde{P}_n\}_{n\geqslant 0}$ is determined by

$$\widetilde{P}_n(x) = d^n \widehat{P}_n\left(\frac{x}{d}\right), \quad n = 0, 1, 2, \ldots, \tag{3.33}$$

where $\{\widehat{P}_n\}_{n\geqslant 0}$ is the sequence of monic OPRL obtained from $\{P_n\}_{n\geqslant 0}$ by

$$\widehat{P}_{nN+j}(x) = \frac{1}{\Lambda_{N-1}(x)}\left\{\Delta_{1,j-1}(x;n)P_{n+1}(\pi_N(x)) + \left(\prod_{i=1}^{j}\widehat{\gamma}_{nN+i}\right)\Delta_{j+1,N-1}(x;n)P_n(\pi_N(x))\right\} \tag{3.34}$$

for all $j = 0, 1, \ldots, N-1$ and $n = 0, 1, 2, \ldots$, and

$$\Lambda_{N-1}(x) := \Delta_{1,N-1}(x;0).$$

Proof. Let $\{\beta_n, \gamma_{n+1}\}_{n\geqslant 0}$ and $\{\widetilde{\beta}_n, \widetilde{\gamma}_{n+1}\}_{n\geqslant 0}$ be the sequences of parameters which appear in the three-term recurrence relations for $\{P_n\}_{n\geqslant 0}$ and $\{\widetilde{P}_n\}_{n\geqslant 0}$, respectively. According to the Geronimus relations (2.6), a straightforward computation yields

$$\widetilde{\beta}_n = d\widehat{\beta}_n, \quad \widetilde{\gamma}_{n+1} = d^2\widehat{\gamma}_{n+1}, \quad n = 0, 1, 2, \ldots, \tag{3.35}$$

where $\{\widehat{\beta}_n, \widehat{\gamma}_{n+1}\}_{n\geqslant 0}$ is the sequence defined by (3.3)–(3.4). Hence, (3.33) holds, where $\{\widehat{P}_n\}_{n\geqslant 0}$ is the sequence of monic OPRL defined by (3.13). We have shown that this sequence $\{\widehat{P}_n\}_{n\geqslant 0}$ satisfies (3.32), hence it is obtained from $\{P_n\}_{n\geqslant 0}$ by an admissible polynomial mapping, π_N being the admissible polynomial. Therefore, taking into account (3.32) and since $\Delta_{1,N-1}(x;n)$ is independent of n, so that $\Delta_{1,N-1}(x;n) = \Lambda_{N-1}(x)$ for all $n = 0, 1, 2, \ldots$, from (3.16) we get the explicit representation (3.34). Thus the theorem is proved. □

Notice that the polynomial π_N is obtained from $\widehat{P}_N$ by adding the constant $\beta_0 = -\Phi_1(0) \equiv \alpha_0$, hence it is not clear that it generates a polynomial mapping in the sense described in [13], since for that π_N must have N real and simple zeros such that the values of π_N calculated in the zeros of the derivative π_N' must be out the true interval of orthogonality $[\xi, \eta]$ of the sequence $\{P_n\}_{n\geqslant 0}$ (which, in our case, is a subset of $[-1, 1]$)—cf. [13, Lemma 1]. Fortunately, Lemma 3.6 in below shows that our polynomial π_N satisfies, in fact, such properties (in fact, it will be shown that π_N is an admissible polynomial). For the proof we first need to state the connection between the sequences of OPRL $\{Q_n\}_{n\geqslant 0}$ and $\{\widetilde{Q}_n\}_{n\geqslant 0}$, where $\{Q_n\}_{n\geqslant 0}$ is constructed from $\{P_n\}_{n\geqslant 0}$ by (2.7)–(2.8) and $\{\widetilde{Q}_n\}_{n\geqslant 0}$ is constructed from $\{\widetilde{P}_n\}_{n\geqslant 0}$ via the same procedure, so that

$$\widetilde{Q}_{n-1}(x) = \frac{1}{x^2-1}\{\widetilde{P}_{n+1}(x) - \widetilde{a}_n^{(1)}\widetilde{P}_n(x) - \widetilde{a}_n^{(0)}\widetilde{P}_{n-1}(x)\} \tag{3.36}$$

for all $n = 1, 2, \ldots$, with

$$\widetilde{a}_n^{(1)} := -\widetilde{\beta}_n + \widetilde{\Phi}_{2n-1}(0), \quad \widetilde{a}_n^{(0)} := \frac{1+\widetilde{\Phi}_{2n}(0)}{1-\widetilde{\Phi}_{2n}(0)}\widetilde{\gamma}_n. \tag{3.37}$$

Theorem 3.4. *Denote by* $\mathrm{d}\sigma$ *and* $\mathrm{d}\widetilde{\sigma}$ *the measures with respect to which* $\{P_n\}_{n\geqslant 0}$ *and* $\{\widetilde{P}_n\}_{n\geqslant 0}$ *are orthogonal, respectively, and let* $\{Q_n\}_{n\geqslant 0}$ *and* $\{\widetilde{Q}_n\}_{n\geqslant 0}$ *be the sequences of monic OPRL with respect to* $(1-x^2)\,\mathrm{d}\sigma$ *and* $(1-x^2)\,\mathrm{d}\widetilde{\sigma}$, *respectively. Then*

$$\widetilde{Q}_{nN-1}(x) = d^{nN-1}\rho_{N-1}\left(\frac{x}{d}\right)Q_{n-1}\left(\pi_N\left(\frac{x}{d}\right)\right) \tag{3.38}$$

for all $n = 1, 2, \ldots$, *where* ρ_{N-1} *is the monic polynomial of degree* $N-1$

$$\rho_{N-1}(x) := [\pi_N^2(x) - 1]/[(x^2-d^{-2})\Lambda_{N-1}(x)]. \tag{3.39}$$

Proof. Taking into account (1.3), (3.35) and (3.3)–(3.4), by (3.37) we find $\widetilde{a}_{nN}^{(1)} = 0$ and $\widetilde{a}_{nN}^{(0)} = d^2\widehat{\gamma}_{nN}a_n^{(0)}/\gamma_n$, hence from (3.36) we can write

$$(d^2x^2-1)\widetilde{Q}_{nN-1}(dx) = d^{nN+1}\widehat{P}_{nN+1}(x) - d^2\frac{\widehat{\gamma}_{nN}}{\gamma_n}a_n^{(0)}d^{nN-1}\widehat{P}_{nN-1}(x)$$

for all $n = 1, 2, \ldots$. Now, we compute $\widehat{P}_{nN+1}(x)$ and $\widehat{P}_{nN-1}(x)$ from Theorem 3.3, so that the above equality can be rewritten as

$$\begin{aligned} &d^{1-nN}\Lambda_{N-1}(x)(x^2-d^{-2})\widetilde{Q}_{nN-1}(dx) \\ &\quad = P_{n+1}(\pi_N(x)) + \widehat{\gamma}_{nN+1}\Delta_{2,N-1}(x;n)P_n(\pi_N(x)) - \widehat{\gamma}_{nN}\gamma_n^{-1}a_n^{(0)} \\ &\qquad \times\left\{\Delta_{1,N-2}(x;n-1)P_n(\pi_N(x)) + \left(\prod_{i=1}^{N-1}\widehat{\gamma}_{(n-1)N+i}\right)P_{n-1}(\pi_N(x))\right\}, \end{aligned}$$

or, taking into account that $\prod_{i=1}^{N-1}\widehat{\gamma}_{(n-1)N+i} = \gamma_n/\widehat{\gamma}_{nN}$, as well as (2.7),

$$\begin{aligned} &d^{1-nN}\Lambda_{N-1}(x)(x^2-d^{-2})\widetilde{Q}_{nN-1}(dx) \\ &\quad = [\pi_N^2(x)-1]Q_{n-1}(\pi_N(x)) \\ &\qquad + \{a_n^{(1)} + \widehat{\gamma}_{nN+1}\Delta_{2,N-1}(x;n) - \widehat{\gamma}_{nN}\gamma_n^{-1}a_n^{(0)}\Delta_{1,N-2}(x;n-1)\}P_n(\pi_N(x)). \end{aligned}$$

Therefore, we see that (3.38) will be proved if we can show that the expression inside {} in this last equality vanishes for all $n = 1, 2, \ldots$, which, according to (2.8), is equivalent to showing that

$$\Phi_{2n-1}(0) - \beta_n + \widehat{\gamma}_{nN+1}\Delta_{2,N-1}(x;n) - \frac{1+\Phi_{2n}(0)}{1-\Phi_{2n}(0)}\widehat{\gamma}_{nN}\Delta_{1,N-2}(x;n-1) = 0 \tag{3.40}$$

for all $n = 1, 2, \ldots$. This will follow from Lemma 3.1. In fact, by (3.21), the left-hand side of (3.40) can be written as

$$\Phi_{2n-1}(0) - \beta_0 + \gamma\Delta_{2,N-1}(x;0) - \frac{2\widehat{\gamma}_{nN}}{1-\Phi_{2n}(0)}\Delta_{1,N-2}(x;n-1),$$

and since the relation $2\widehat{\gamma}_{nN}/[1-\Phi_{2n}(0)] = \gamma$ holds, we conclude from (3.22) that the last expression vanishes. $\square$

Remark 3.5. If $b_1 = b_2 = \cdots = b_{N-1} = 0$, straightforward computations give

$$d = 2^{(1-N)/N}, \quad \pi_N(x) = T_N(dx), \quad \Lambda_{N-1}(x) = \rho_{N-1}(x) = dU_{N-1}(dx)$$

(T_N and U_{N-1} being the classical Chebyshev polynomials of the first and second kind, resp.), hence (3.32) and (3.38) generalize formulas (3.7) in [19]. Moreover, for general $b_1, \ldots, b_{N-1} \in (-1, 1)$, one see that there is a one-to-one correspondence between the family $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ of OPUC and the two families $\{\widetilde{P}_n\}_{n\geqslant 0}$ and $\{\widetilde{Q}_n\}_{n\geqslant 0}$ of OPRL on (a subset of) the interval $[-1, 1]$.

Lemma 3.6. *If $z_1, z_2, \ldots, z_{N-1}$ denote the zeros of Λ_{N-1} then, up to permutations,*

$$-\frac{1}{d} \equiv z_0 < z_1 < z_2 < \cdots < z_{N-1} < z_N \equiv \frac{1}{d}, \tag{3.41}$$

and for this ordering

$$\pi_N(z_i) = (-1)^{N-i}, \quad i = 0, 1, \ldots, N. \tag{3.42}$$

As a consequence, π_N has N real and simple zeros, all in $]-\frac{1}{d}, \frac{1}{d}[$ and interlacing with those of Λ_{N-1}, and π'_N has $N-1$ real and simple zeros $y_1 < y_2 < \cdots < y_{N-1}$ such that

$$|\pi_N(y_i)| \geqslant 1, \quad i = 1, \ldots, N-1.$$

Proof. By definition, $\Lambda_{N-1} \equiv \widehat{P}^{(1)}_{N-1}$ is an element of the OP family $\{\widehat{P}^{(1)}_n\}_{n\geqslant 0}$, hence Λ_{N-1} has $N-1$ real and simple zeros. Furthermore, the true interval of orthogonality of $\{\widehat{P}^{(1)}_n\}_{n\geqslant 0}$ is a subset of the true interval of orthogonality of $\{\widehat{P}_n\}_{n\geqslant 0}$ [9, p. 87]. Therefore, taking into account that $\widehat{P}_n(x) = d^{-n}\widetilde{P}_n(dx)$ and that the true interval of orthogonality of $\{\widetilde{P}_n\}_{n\geqslant 0}$ is a subset of $[-1, 1]$, it follows that all the zeros of Λ_{N-1} belong to the interval $]-1/d, 1/d[$. This justifies (3.41). In order to prove (3.42), put $n = 1$ in (3.38) to get

$$\pi^2_N(x) - 1 = d^{1-N}(x^2 - d^{-2})\Lambda_{N-1}(x)\widetilde{Q}_{N-1}(dx), \tag{3.43}$$

which implies

$$|\pi_N(z_i)| = 1, \quad i = 0, 1, \ldots, N. \tag{3.44}$$

By (3.8) and (4.4) in [9, p. 86], we find $\widehat{P}^{(1)}_N(x)\widehat{P}_N(x) - \widehat{P}_{N+1}(x)\widehat{P}^{(1)}_{N-1}(x) = \gamma_1 > 0$. In this equality set $x = z_i$ and $x = z_{i+1}$ for an arbitrary but fixed $i \in \{1, \ldots, N-2\}$, to get $\widehat{P}_N(z_i)\widehat{P}_N(z_{i+1}) = \gamma_1^2/\widehat{P}^{(1)}_N(z_i)\widehat{P}^{(1)}_N(z_{i+1}) < 0$, the last inequality being justified by the separation theorem for the zeros of an orthogonal polynomial system [9, p. 28]. Therefore, $\widehat{P}_N(z_i)$ and $\widehat{P}_N(z_{i+1})$ have opposite signs for $i = 1, 2, \ldots, N-2$. This still is true for $i = 0$ and $i = N-1$. In fact, for $i = N-1$ the statement follows from $\widehat{P}_N(z_N) = \widehat{P}_N(1/d) > 0$ and $\widehat{P}_N(z_{N-1}) = \gamma_1/\widehat{P}^{(1)}_N(z_{N-1}) < 0$; and for $i = 0$ we see that $\widehat{P}_N(z_0) = \widehat{P}_N(-1/d)$ has the same sign as $(-1)^N$, and $\widehat{P}_N(z_1) = \gamma_1/\widehat{P}^{(1)}_N(z_1)$ has the opposite sign of $(-1)^N$. We conclude that $\widehat{P}_N(z_i)$ and $\widehat{P}_N(z_{i+1})$ have opposite signs for $i = 0, 1, \ldots, N-1$, hence

$$\pi_N(z_i) - \pi_N(z_{i+1}) = \widehat{P}_N(z_i) - \widehat{P}_N(z_{i+1}) \neq 0, \quad i = 0, 1, \ldots, N-1.$$

This together with (3.44) implies $\pi_N(z_i)\pi_N(z_{i+1}) = -1$ for all $i = 0, 1, \ldots, N-1$, and since $\pi_N(z_N) = 1$ (which follows from $\pi_N(z_N) = \widehat{P}_N(1/d) + \beta_0 > -1$, this last inequality being justified by $\widehat{P}_N(1/d) > 0$ and the basic fact $\beta_0 \in]\xi, \eta[\subset] - 1, 1[$), (3.42) is proved. The remaining statements in the Lemma are now obvious. $\square$

Remark 3.7. By definition, the three fundamental polynomials π_N, Λ_{N-1}, and ρ_{N-1} depend only on the choice of $\alpha_0, b_1, \ldots, b_{N-1}$. Fig. 1 shows a plot of these polynomials in the special case $N = 5$ and for the choice $\alpha_0 = -\frac{1}{2}$, $b_1 = \frac{1}{2}$, $b_2 = \frac{1}{3}$, $b_3 = -\frac{1}{4}$ and $b_4 = \frac{1}{8}$. Furthermore, it follows from the previous Lemma that $\pi_N^{-1}([-1, 1])$ is the union of N intervals of the real line contained in the interval $[-1/d, 1/d]$, such that two of these intervals have at most one common point. In the graphic we can see the five intervals that form the set $\pi_N^{-1}([-1, 1])$ in the example under consideration (notice that $1/d \simeq 1.91874$).

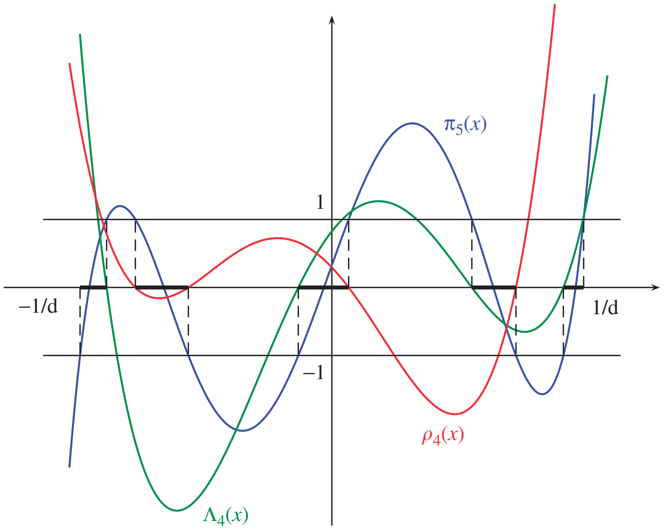

Fig. 1.

3.3. Relation between the measures on the real line

The previous Lemma and Theorem 3.3 show that, in fact, $\pi_N(x)$ is an admissible polynomial mapping. Thus, the following proposition holds.

Theorem 3.8. *Under the conditions of Theorem* 3.3, *let* $\mathrm{d}\sigma$, $\mathrm{d}\widetilde{\sigma}$, *and* $\mathrm{d}\widehat{\sigma}$ *denote the measures associated with the sequences of monic OPRL* $\{P_n\}_{n\geqslant 0}$, $\{\widetilde{P}_n\}_{n\geqslant 0}$, *and* $\{\widehat{P}_n\}_{n\geqslant 0}$, *respectively*, *and let* $[\xi,\eta]$ *be the true interval of orthogonality of* $\{P_n\}_{n\geqslant 0}$ (*with* $-1\leqslant\xi<\eta\leqslant 1$). *Then the following statements hold*:

(i) *The Stieltjes transforms of* $\mathrm{d}\widetilde{\sigma}$ *and* $\mathrm{d}\sigma$ *are related by*

$$F(z;\mathrm{d}\widetilde{\sigma})=\frac{1}{d}\Lambda_{N-1}\left(\frac{z}{d}\right)F\left(\pi_N\left(\frac{z}{d}\right);\mathrm{d}\sigma\right),\qquad \frac{z}{d}\in\mathbb{C}\backslash\pi_N^{-1}([\xi,\eta]). \tag{3.45}$$

(ii) *Up to constant factors*, *the relation between the measures is given by*

$$\mathrm{d}\widetilde{\sigma}(x)=\mathrm{d}\widehat{\sigma}\left(\frac{x}{d}\right)=\frac{\left|\Lambda_{N-1}\left(\frac{x}{d}\right)\right|}{\pi_N'\left(\frac{x}{d}\right)}\mathrm{d}\sigma\left(\pi_N\left(\frac{x}{d}\right)\right),\qquad \frac{x}{d}\in\pi_N^{-1}([\xi,\eta]), \tag{3.46}$$

so that $\mathrm{supp}(\mathrm{d}\widetilde{\sigma})$ *is a subset of* $[-1,1]$ *contained in the union of N intervals.*

Proof. In order to prove (3.45), we first show that

$$\widehat{P}^{(1)}_{nN-1}(x)=\Lambda_{N-1}(x)P^{(1)}_{n-1}(\pi_N(x)),\qquad n=0,1,2,\ldots \tag{3.47}$$

(this relation is expected, according to the results in [13]). We know the three-term recurrence relation (3.1) for the P_n's, also satisfied by the numerator polynomials $P^{(1)}_{n-1}$'s, with $P^{(1)}_{-1}\equiv 0$ and $P^{(1)}_0\equiv 1$. According to (3.32), the contraction of the three-term recurrence relation for the $\widehat{P}_n$'s must yield (3.1) with argument $\pi_N(x)$. The numerator polynomials $\widehat{P}^{(1)}_{n-1}$ satisfy the same three-term recurrence relation as $\widehat{P}_n$, so $\widehat{P}^{(1)}_{nN-1}$ must be a linear combination of the two solutions of (3.1) with argument $\pi_N(x)$, hence

$$\widehat{P}^{(1)}_{nN-1}(x)=\alpha(x)P_n(\pi_N(x))+\beta(x)P^{(1)}_{n-1}(\pi_N(x)),\qquad n=0,1,2,\ldots$$

where $\alpha(x)$ and $\beta(x)$ are two polynomials independent of n. But, for $n=0$ we see that $\alpha(x)\equiv 0$, and with $n=1$ we find $\beta(x)=\widehat{P}^{(1)}_{N-1}(x)=\Lambda_{N-1}(x)$, and thus (3.47) is proved. Now, by (3.47) and Markov's Theorem, it is easy to state the relation between the Stieltjes transforms. Indeed, since we can always assume that $\widehat{u}_0 = u_0 := \int_{\mathrm{supp}(\mathrm{d}\sigma)} \mathrm{d}\sigma(y)$, we get

$$F(z;\mathrm{d}\widehat{\sigma}) = -\widehat{u}_0 \lim_{n\to+\infty} \frac{\widehat{P}^{(1)}_{nN-1}(z)}{\widehat{P}_{nN}(z)} = -u_0 \Lambda_{N-1}(z) \lim_{n\to+\infty} \frac{P^{(1)}_{n-1}(\pi_N(z))}{P_n(\pi_N(z))}$$
$$= \Lambda_{N-1}(z) F(\pi_N(z);\mathrm{d}\sigma), \quad z \notin \pi_N^{-1}([\xi,\eta]);$$

hence

$$F(z;\mathrm{d}\widetilde{\sigma}) = \frac{1}{d} F\left(\frac{z}{d};\mathrm{d}\widehat{\sigma}\right) = \frac{1}{d}\Lambda_{N-1}\left(\frac{z}{d}\right) F\left(\pi_N\left(\frac{z}{d}\right);\mathrm{d}\sigma\right), \quad \frac{z}{d} \notin \pi_N^{-1}([\xi,\eta]).$$

We now prove (3.46). According to Lemma 3.6, one can write

$$\pi_N^{-1}([\xi,\eta]) = \bigcup_{i=1}^{N} J_i \subset \left[-\frac{1}{d},\frac{1}{d}\right],$$

where $J_1,\dots,J_N$ are N closed intervals in the real line (disposed from the left to the right) such that J_i and J_{i+1} have at most one common point. Let $D_1 := [-1/d, y_1]$, $D_i := [y_{i-1}, y_i]$ $(i = 2,\dots,N-1)$, $D_N := [y_{N-1}, 1/d]$, and consider the functions $T_i : D_i \to \pi_N(D_i)$ $(i = 1,\dots,N)$ defined by $T_i(x) := \pi_N(x)$ for $x \in D_i$. Then, each T_i is bijective and $T_i(J_i) = [\xi,\eta]$ for $i = 1,\dots,N$. We know that $[\xi,\eta] \subset [-1,1]$ and we also have (3.42). Hence $\pi_N(z_i) \notin]\xi,\eta[$ $i = 1,\dots,N-1$), and we see that z_i is located between the intervals J_i and J_{i+1} (z_i may be the right-hand point of J_i or the left-hand point of J_{i+1}). We first prove that, under the hypothesis of the Theorem, the right-hand side of (3.46) defines, in fact, a measure with finite moments. For that, it is sufficient to show that

$$\int_{\pi_N^{-1}([\xi,\eta])} |\Lambda_{N-1}(x)| \frac{\mathrm{d}\sigma(\pi_N(x))}{\pi_N'(x)} < +\infty \tag{3.48}$$

(because $[\xi,\eta]$ is compact). Notice that $|\pi_N(y_i)| \geqslant 1$ for all $i = 1,\dots,N-1$. Hence, if $|\pi_N(y_i)| > 1$ for all $i = 1,\dots,N-1$ or if $-1 < \xi < \eta < 1$ then it is clear that (3.48) holds. However, if $\pi_N(y_i) = \xi = -1$ or $\pi_N(y_i) = \eta = 1$ for some i then from Lemma 3.6 we see that necessarily $y_i = z_i$, hence $\Lambda_{N-1}(x)/\pi_N'(x)$ has no pole at $x = y_i$, and we can conclude that

$$\sup_{x\in\pi_N^{-1}(]\xi,\eta[)} \left|\frac{\Lambda_{N-1}(x)}{\pi_N'(x)}\right| < \infty,$$

and (3.48) also follows. Now, for $z \in \mathbb{C}\backslash\pi_N^{-1}([\xi,\eta])$, noticing that Λ_{N-1} and π_N' have the same sign in each interval J_i, we get

$$\int_{\pi_N^{-1}([\xi,\eta])} \frac{1}{x-z} |\Lambda_{N-1}(x)| \frac{\mathrm{d}\sigma(\pi_N(x))}{\pi_N'(x)}$$
$$= \sum_{i=1}^{N} \int_{J_i} \frac{1}{x-z} \frac{\Lambda_{N-1}(x)}{\pi_N'(x)} \operatorname{sgn}\{\pi_N'(x)\}\, \mathrm{d}\sigma(\pi_N(x))$$
$$= \int_{\xi}^{\eta} \sum_{i=1}^{N} \frac{1}{T_i^{-1}(y)-z} \frac{\Lambda_{N-1}(T_i^{-1}(y))}{\pi_N'(T_i^{-1}(y))}\, \mathrm{d}\sigma(y),$$

the last identity follows from the substitutions $y = \pi_N(x)$, $x \in J_i$ $(i = 1,\dots,N)$. Therefore, since decomposition into partial fractions gives

$$\frac{\Lambda_{N-1}(z)}{y-\pi_N(z)} = \sum_{i=1}^{N} \frac{1}{T_i^{-1}(y)-z} \frac{\Lambda_{N-1}(T_i^{-1}(y))}{\pi_N'(T_j^{-1}(y))},$$

it follows that, for $z \in \mathbb{C}\backslash\pi_N^{-1}([\xi,\eta])$,

$$\int_{\pi_N^{-1}([\xi,\eta])} \frac{1}{x-z}|\Lambda_{N-1}(x)|\frac{\mathrm{d}\sigma(\pi_N(x))}{\pi_N'(x)} = \Lambda_{N-1}(z)F(\pi_N(z);\mathrm{d}\sigma) = F(z;\mathrm{d}\widehat{\sigma}),$$

which proves (3.46). □

4. Solution to Problem (P)

We are assuming that the Verblunsky coefficients for $\{\Phi_n(z)\}_{n\geqslant 0}$ are real and satisfy $-1<\Phi_n(0)<1$ for all $n=1,2,\ldots$. This is equivalent to say that $\mathrm{d}\mu$ is symmetric, in the usual sense that $\mathrm{d}\mu(\theta)+\mathrm{d}\mu(2\pi-\theta)=0$, which means that

$$\int_0^{2\pi} f(\theta)\,\mathrm{d}\mu(\theta) + \int_0^{2\pi} f(\theta)\,\mathrm{d}\mu(2\pi-\theta) = 0$$

for every $f\in\mathscr{C}[0,2\pi]$. Then $\theta\in\operatorname{supp}(\mathrm{d}\mu)$ if and only if $2\pi-\theta\in\operatorname{supp}(\mathrm{d}\mu)$, and so one can write

$$\operatorname{supp}(\mathrm{d}\mu) \subset [\theta_m,\theta_M]\cup[2\pi-\theta_M,2\pi-\theta_m], \tag{4.1}$$

$$\theta_m := \min_{0\leqslant\theta\leqslant\pi} \operatorname{supp}(\mathrm{d}\mu), \quad \theta_M := \max_{0\leqslant\theta\leqslant\pi} \operatorname{supp}(\mathrm{d}\mu). \tag{4.2}$$

Under these conditions, since $\mathbf{b}\equiv(b_1,\ldots,b_{N-1})$ is a finite sequence of real numbers belonging to the interval $(-1,1)$ then the measure with respect to which $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ is orthogonal is also symmetric. In order to find this measure it is convenient to introduce some notation and preliminary results.

4.1. Notation

We set

$$\mathbb{C}^* := \mathbb{C}\backslash\{0\}, \quad \partial\mathbb{D}^- := \{z\in\mathbb{C} : |z|=1, \Im(z)\leqslant 0\}.$$

Define

$$h(z) := z-\sqrt{z^2-1}, \quad z\in\mathbb{C}, \tag{4.3}$$

by choosing the branch of the square root so that $|h(z)| = |z-\sqrt{z^2-1}|<1$ for $z\in\mathbb{C}\backslash[-1,1]$. Then h is analytic on the complex plane with a cut along the real interval $[-1,1]$, with expansion

$$h(z) = \frac{1}{2}\frac{1}{z}+\frac{1}{8}\frac{1}{z^3}+\cdots, \quad |z|>1. \tag{4.4}$$

We also define

$$g(z) := \frac{1}{2d}\left(z+\frac{1}{z}\right), \tag{4.5}$$

which is an analytic function on $\mathbb{C}^*$ that maps $\partial\mathbb{D}$ onto the real interval $[-1/d,1/d]$ and maps $\mathbb{C}^*\backslash\partial\mathbb{D}$ onto $\mathbb{C}\backslash[-1/d, 1/d]$. Further, we set

$$E_{\mathbf{b}} := \left\{\theta\in[0,2\pi] : \frac{1}{d}\cos\theta \in \pi_N^{-1}([-1,1])\right\}, \tag{4.6}$$

$$\Gamma_{\mathbf{b}} := \{\mathrm{e}^{\mathrm{i}\theta} : \theta\in E_{\mathbf{b}}\}. \tag{4.7}$$

By Lemma 3.6 we can write

$$\pi_N^{-1}([-1,1]) = \bigcup_{\ell=1}^{N} I_\ell,$$

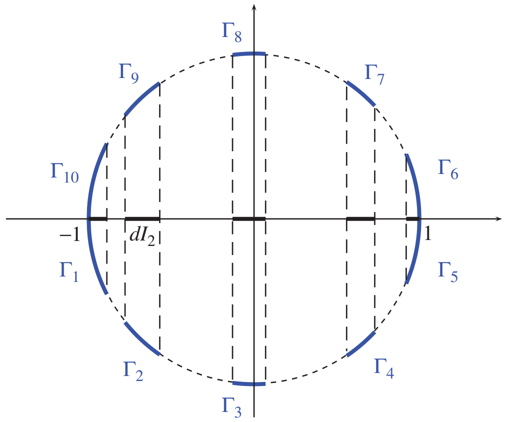

Fig. 2.

where $I_1, \ldots, I_N$ are N closed intervals on the real line (disposed from the left to the right) such that I_ℓ and $I_{\ell+1}$ have at most one common point and $\pi_N(I_\ell) = [-1, 1]$ for every ℓ. Then we see that $\Gamma_{\mathbf{b}}$ is the union of $2N$ arcs on the unit circle,

$$\Gamma_{\mathbf{b}} \equiv \bigcup_{\ell=1}^{2N} \Gamma_\ell,$$

so that to the (rescaled) interval dI_ℓ corresponds the two symmetric arcs (with respect to the real axis) Γ_ℓ and $\Gamma_{2N+1-\ell}$, for each $\ell = 1, \ldots, N$ (see Fig. 2 for an illustrative example with $N = 5$ and the same data as in Fig. 1—notice that the five intervals appearing in Fig. 1 are $I_1, I_2, \ldots, I_5$, while in Fig. 2 these were rescaled, so that the five intervals on the real line appearing in Fig. 2 are $dI_1, dI_2, \ldots, dI_5$, originating 10 arcs $\Gamma_1, \Gamma_2, \ldots, \Gamma_{10}$). Finally, set

$$E_\ell := \{\theta \in [0, 2\pi] : e^{i\theta} \in \Gamma_\ell\} \quad (\ell = 1, 2, \ldots, 2N), \tag{4.8}$$

so that $E_1, \ldots, E_{2N}$ are $2N$ intervals contained in $[0, 2\pi]$ such that E_ℓ and $E_{\ell+1}$ have at most one common point, and

$$E_{\mathbf{b}} = \bigcup_{\ell=1}^{2N} E_\ell.$$

Lemma 4.1. *Fix $z \in \mathbb{C}^*$. Then*

(i) *$z \in \Gamma_{\mathbf{b}}$ if and only if $\pi_N(g(z)) \in [-1, 1]$.*
(ii) *If $z \in \Gamma_{\mathbf{b}}$ then $\rho_{N-1}(g(z))\Lambda_{N-1}(g(z)) \in [0, +\infty[$.*
(iii) *If $z \in \partial\mathbb{D} \backslash \Gamma_{\mathbf{b}}$ then $\rho_{N-1}(g(z))\Lambda_{N-1}(g(z)) \in]-\infty, 0[$.*
(iv) *If z belongs to the punctured disc $0 < |z| < 2 - \sqrt{3}$ or to the annulus $|z| > 2 + \sqrt{3}$, then $|\pi_N(g(z))| \geqslant 1/d^N > 1$.*

Proof. Fix $z \neq 0$. If $z \in \Gamma_{\mathbf{b}}$ then $z = e^{i\theta}$ for some $\theta \in E_{\mathbf{b}}$, hence $g(z) = (1/d)\cos\theta$ and by definition of $E_{\mathbf{b}}$ we get $\pi_N(g(z)) \in [-1, 1]$. Assume now $\pi_N(g(z)) \in [-1, 1]$. Put $z = \rho e^{i\theta}$, with $\rho > 0$ and $\theta \in [0, 2\pi[$. We first show that $\rho = 1$. Assume $\rho \neq 1$. Since

$$g(z) = \frac{\rho + \rho^{-1}}{2d}\cos\theta + i\frac{\rho - \rho^{-1}}{2d}\sin\theta,$$

we see that $g(z)$ lies over the ellipse whose major and minor axes lie on the real and imaginary axes of the complex plane (resp.), the lengths of the major and minor axes being equal to $(\rho + \rho^{-1})/2d$ and $|\rho - \rho^{-1}|/2d$ (resp.), with focal points at $\pm 1/d$, independent of ρ. This implies $g(z) \notin [-1/d, 1/d]$. Hence, if $\sin\theta = 0$ then $g(z)$ is a real number outside the interval $[-1/d, 1/d]$ and so from Lemma 3.6 we see that $\pi_N(g(z)) \notin [-1, 1]$, which is a contradiction with the hypothesis. If, however, $\sin\theta \neq 0$ let us show that, again, $\pi_N(g(z)) \notin [-1, 1]$. In fact, suppose $\sin\theta \neq 0$ and $\pi_N(g(z)) = \ell \in [-1, 1]$. Then $g(z)$ would be a complex non-real zero of the polynomial t_N, of degree N and with real coefficients, defined by

$$t_N(x) := \ell - \pi_N(x).$$

But, this is impossible. Indeed, if $-1 < \ell < 1$ then by Lemma 3.6 we find

$$t_N(z_i)t_N(z_{i+1}) = [\ell - (-1)^{N-i}] \cdot [\ell - (-1)^{N-i-1}] < 0$$

for every $i = 0, 1, \ldots, N-1$, so all the zeros of t_N are real (and simple, one in each interval $]z_i, z_{i+1}[$). If $\ell = \pm 1$, then $g(z)$ would be a zero of the polynomial on the right-hand side of (3.43), which is impossible since Λ_{N-1} and $\widetilde{Q}_{N-1}$ belong to orthogonal families and so all their zeros are real. We conclude that $\rho = 1$. Therefore, $z = \mathrm{e}^{\mathrm{i}\theta}$, hence $g(z) = (1/d)\cos\theta$, so that $\pi_N((1/d)\cos\theta) = \pi_N(g(z)) \in [-1, 1]$ (by hypothesis), which implies $\theta \in E_{\mathbf{b}}$, so $z \in \Gamma_{\mathbf{b}}$. This proves (i). Now, by (3.39) we can write

$$\pi_N^2(g(z)) - 1 = -\frac{\sin^2\theta}{d^2}\rho_{N-1}(g(z))\Lambda_{N-1}(g(z)), \quad z = \mathrm{e}^{\mathrm{i}\theta} \in \partial\mathbb{D},$$

from which (ii) and (iii) follows, taking into account (i) and the fact that $\pi_N(g(z))$ is a real number for $z \in \partial\mathbb{D}$. In order to prove (iv), put $z = \rho\mathrm{e}^{\mathrm{i}\theta}$, with $\rho > 0$. If $\sin\theta = 0$ the statement in (iv) comes from (i). Assume $\sin\theta \neq 0$. According to Lemma 3.6, there exists N real distinct numbers $r_1, \ldots, r_N$, with $-1 < r_1 < \cdots < r_N < 1$, such that $r_1/d, \ldots, r_N/d$ are the N distinct zeros of π_N. Then

$$\pi_N(g(z)) = \frac{1}{d^N}\prod_{j=1}^{N}\left(\frac{1}{2}(z + z^{-1}) - r_j\right).$$

Put $r_j \equiv \cos\alpha_j$ with $-\pi < \alpha_j < 0$ $(1 \leqslant j \leqslant N)$, and let η be that positive real number such that $\cosh\eta = \frac{1}{2}(\rho + \rho^{-1})$. It is straightforward to verify that

$$\left|\frac{1}{2}(z + z^{-1}) - r_j\right|^2 = [\cosh\,\eta - \cos(\theta - \alpha_j)] \cdot [\cosh\,\eta - \cos(\theta + \alpha_j)]$$

for all $j = 1, \ldots, N$. Therefore, if $\cosh\,\eta \geqslant 2$, i.e., if $\rho \notin]2 - \sqrt{3}, 2 + \sqrt{3}[$, we have $|\pi_N(g(z))| \geqslant 1/d^N > 1$. $\square$

4.2. The fundamental mappings Θ and K

In the next lemmas we introduce two mappings, Θ and K, which play a fundamental role in the description of the perturbation process on the unit circle defined by Problem (**P**). We notice that these lemmas are motivated by the lecture of the paper [30] by Peherstorfer and Steinbauer.

Lemma 4.2. *If g and h the mappings defined in* (4.5) *and* (4.3), *set*

$$\Theta(z) := \begin{cases} h \circ \pi_N \circ g(z) & \text{if } z \in \mathbb{C}^*, \\ 0 & \text{if } z = 0, \end{cases} \tag{4.9}$$

and let $E_{\mathbf{b}}$, $\Gamma_{\mathbf{b}}$, Γ_ℓ and E_ℓ $(\ell = 1, \ldots, 2N)$ as in (4.6)–(4.8). *Then*

(i) *Θ is analytic on $\mathbb{C}\backslash\Gamma_{\mathbf{b}}$ and maps this set into $\mathbb{D}$.*
(ii) *$|\Theta(\mathrm{e}^{\mathrm{i}\theta})| = 1$ for $\theta \in E_{\mathbf{b}}$; and $\Theta(\mathrm{e}^{\mathrm{i}\theta}) \in (-1, 1)$ for $\theta \in [0, 2\pi]\backslash E_{\mathbf{b}}$.*
(iii) *For every $\ell = 1, \ldots, 2N$, the mapping $\Theta|\Gamma_\ell : \Gamma_\ell \to \partial\mathbb{D}^-$ is continuous and bijective, so*

$$\Theta(\Gamma_\ell) = \partial\mathbb{D}^-, \quad \ell = 1, \ldots, 2N.$$

(iv) *Let* $\vartheta_N : E_{\mathbf{b}} \to [0, \pi]$ *be defined by*

$$\vartheta_N(\theta) = -\mathrm{Arg}(\Theta(\mathrm{e}^{\mathrm{i}\theta})), \quad \theta \in E_\ell^0 \quad (\ell = 1, \ldots, 2N) \tag{4.10}$$

where Arg *denotes the principal argument. Then* $\vartheta_N|E_\ell : E_\ell \mapsto [0, \pi]$ *is bijective and continuous at all points of* E_ℓ *up to the end points, for all* $\ell = 1, \ldots, 2N$, *and*

$$\cos \vartheta_N(\theta) = \pi_N\left(\frac{1}{d}\cos\theta\right), \quad \theta \in E_{\mathbf{b}}. \tag{4.11}$$

(v) $\Theta(z) = 0$ *if and only if* $z = 0$. *Moreover*, 0 *is a zero of order* N *of* Θ *and*

$$\lim_{z\to 0} \frac{\Theta(z)}{z^N} = \frac{(2d)^N}{2} = \prod_{j=1}^{N-1}(1 - b_j^2). \tag{4.12}$$

Proof. We know that g is analytic on $\mathbb{C}^*$ and h is analytic on $\mathbb{C}\backslash[-1, 1]$. Thus, since in $\mathbb{C}^*$ we have $\Theta = h \circ \pi_N \circ g$, it follows that Θ is analytic for all $z \in \mathbb{C}^*$ such that $\pi_N(g(z)) \in \mathbb{C}\backslash[-1, 1]$. Using the fact that $|\Theta(z)| < 1$ for $z \in \mathbb{C}^*\backslash\Gamma_{\mathbf{b}}$ we see that $z = 0$ is a removable singularity, so Θ is also analytic at $z = 0$. Furthermore, according to (4.4) and (iv) in Lemma 4.1, we have

$$\frac{\Theta(z)}{z^N} = \frac{h\,(\pi_N(g(z)))}{z^N} = \frac{1}{2}\frac{1}{z^N \pi_N(g(z))} + \frac{1}{8}\frac{1}{z^N \pi_N^3(g(z))} + \cdots,$$

from which (4.12) follows. To conclude the proof of part (i) note that, by Lemma 4.1, if $z \in \mathbb{C}^*\backslash\Gamma_{\mathbf{b}}$ then $\pi_N(g(z)) \notin [-1, 1]$, hence we find $|\Theta(z)| = |h(\pi_N(g(z)))| < 1$, so that Θ maps $\mathbb{C}\backslash\Gamma_{\mathbf{b}}$ into $\mathbb{D}$, the (geometric) interior of the unit circle.

To prove (ii) notice that if $\theta \in [0, 2\pi]$ then by the choice of the branch of the complex square root in the definition of h, we have

$$\Theta(\mathrm{e}^{\mathrm{i}\theta}) = \begin{cases} \pi_N\left(\frac{1}{d}\cos\theta\right) - \sqrt{\pi_N^2\left(\frac{1}{d}\cos\theta\right) - 1} & \text{if } \pi_N\left(\frac{1}{d}\cos\theta\right) > 1, \\ \pi_N\left(\frac{1}{d}\cos\theta\right) - \mathrm{i}\sqrt{1 - \pi_N^2\left(\frac{1}{d}\cos\theta\right)} & \text{if } \pi_N\left(\frac{1}{d}\cos\theta\right) \in [-1, 1], \\ \pi_N\left(\frac{1}{d}\cos\theta\right) + \sqrt{\pi_N^2\left(\frac{1}{d}\cos\theta\right) - 1} & \text{if } \pi_N\left(\frac{1}{d}\cos\theta\right) < -1 \end{cases} \tag{4.13}$$

(on the right-hand side all the square roots are the ordinary real square roots). Therefore, if $\theta \in E_{\mathbf{b}}$ we have $\pi_N(\frac{1}{d}\cos\theta) \in [-1, 1]$, hence $|\Theta(\mathrm{e}^{\mathrm{i}\theta})| = 1$. If $\theta \in [0, 2\pi]\backslash E_{\mathbf{b}}$ then either $\pi_N(\frac{1}{d}\cos\theta) > 1$ or $\pi_N(\frac{1}{d}\cos\theta) < -1$ and so, taking into account that $x - \sqrt{x^2 - 1} \in]-1, 1[$ for $x > 1$, it follows that $\Theta(\mathrm{e}^{\mathrm{i}\theta}) \in]-1, 1[$ in either cases.

Next we prove (iii). Fix $\ell \in \{1, \ldots, 2N\}$. For each $z \in \Gamma_\ell$ we can write $z = \mathrm{e}^{\mathrm{i}\theta}$ with either $-\pi \leqslant \theta \leqslant 0$ and $\frac{1}{d}\cos\theta \in I_\ell$ if $\ell \in \{1, \ldots, N\}$, or $0 \leqslant \theta \leqslant \pi$ and $\frac{1}{d}\cos\theta \in I_{2N-\ell}$ if $\ell \in \{N+1, \ldots, 2N\}$. In either cases, by the computations above, we have

$$\Theta(z) = \pi_N\left(\frac{1}{d}\cos\theta\right) - \mathrm{i}\sqrt{1 - \pi_N^2\left(\frac{1}{d}\cos\theta\right)}, \quad z = \mathrm{e}^{\mathrm{i}\theta} \in \Gamma_\ell, \tag{4.14}$$

from which the continuity of $\Theta|\Gamma_\ell$ follows. Since $\pi_N(I_\ell) = [-1, 1]$, (4.14) also gives $\Theta(\Gamma_\ell) = \partial\mathbb{D}^-$ and so $\Theta|\Gamma_\ell : \Gamma_\ell \to \partial\mathbb{D}^-$ is onto. It is easy to see that Θ is also one-to-one.

To prove (iv) notice that, according to (iii), we can write

$$\Theta(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{-\mathrm{i}\vartheta_N(\theta)}, \quad \theta \in E_{\mathbf{b}} \tag{4.15}$$

where $\vartheta_N : E_{\mathbf{b}} \to [0, \pi]$ can be chosen such that $\vartheta_N|E_\ell : E_\ell \mapsto [0, \pi]$ is bijective and continuous at all points of E_ℓ up to one of the extreme points, for all $\ell = 1, \ldots, 2N$. It is clear that this mapping ϑ_N can be explicitly given by (4.10) and, further, by (4.15) and (4.14) it also satisfies (4.11).

Finally, the relation $(z-\sqrt{z^2-1})(z+\sqrt{z^2-1})=1$ shows that $h(z) \neq 0$ for all $z \in \mathbb{C}$ and since $\Theta(z)=h(\pi_N(g(z)))$ for $z \neq 0$, this implies that Θ has no zeros different from 0. This completes the proof of (v). $\square$

Remark 4.3. The previous lemma shows that $\Theta(z) = \mathcal{O}(z^N)$ as $z \to 0$, so that the behavior of $\Theta(z)$ near the origin is similar to that of z^N.

In the sequel denote by $\mathrm{Int}(\Gamma_\ell)$ the arc Γ_ℓ without the end points, and set

$$\mathrm{Int}(\Gamma_{\mathbf{b}}) := \bigcup_{\ell=1}^{2N} \mathrm{Int}(\Gamma_\ell).$$

These sets are the analogue of certain arcs described in [30]. We also define $-\pi \equiv \theta_0 < \theta_1 < \cdots < \theta_{N-1} < \theta_N \equiv 0$ by

$$\frac{1}{d}\cos\theta_j := z_j, \quad j = 0, 1, \ldots, N$$

where $z_1, \ldots, z_{N-1}$ are the zeros of Λ_{N-1} (cf. Lemma 3.6). Hence

$$\Lambda_{N-1}(g(z)) = 0 \quad \text{iff } z = \mathrm{e}^{\pm \mathrm{i}\theta_j}, \quad j = 1, 2, \ldots, N-1.$$

Notice that each one of all these points $\mathrm{e}^{\pm\mathrm{i}\theta_j}$ $(j = 0, 1, \ldots, N)$ is an extreme point on some arc Γ_ℓ, hence all these points belong to $\Gamma_{\mathbf{b}}$. Notice also that the totality of the extreme points of the arcs Γ_ℓ $(\ell = 1, 2, \ldots, 2N)$ is given by the set of points $\mathrm{e}^{\pm\mathrm{i}\theta_j}$ $(j=0, 1, \ldots, N)$ together with the points $\mathrm{e}^{\pm\mathrm{i}\zeta_j}$ $(j=1, \ldots, N-1)$, where $-\pi < \zeta_1 < \cdots < \zeta_{N-1} < 0$ are defined by

$$\frac{1}{d}\cos\zeta_j = w_j, \quad j = 1, \ldots, N-1,$$

$w_1 < \cdots < w_{N-1}$ being the zeros of ρ_{N-1} (which also are all real and simple and belong to the open interval $]-1/d, 1/d[$). It follows that

$$\rho_{N-1}(g(z)) = 0 \quad \text{iff} \quad z = \mathrm{e}^{\pm\mathrm{i}\zeta_j}, \quad j = 1, 2, \ldots, N-1.$$

Next we put

$$u_{N-1}(z) := \frac{\Theta(z) - 1/\Theta(z)}{z - 1/z}, \quad z \neq 0, \pm 1.$$

By Lemma 4.2, $u_{N-1}(z)$ becomes an analytic function in $\mathbb{C}^*\backslash\Gamma_{\mathbf{b}}$. Further, from (3.39),

$$d^2 u_{N-1}^2(z) = \rho_{N-1}(g(z))\Lambda_{N-1}(g(z)), \quad z \neq 0, \pm 1.$$

Lemma 4.4. *Define* $K : \mathbb{C}\backslash\{\mathrm{e}^{\pm\mathrm{i}\theta_j}\}_{j=0}^N \to \mathbb{C}$ *by*

$$K(z) := \begin{cases} d u_{N-1}(z)/\Lambda_{N-1}(g(z)) & \text{if } z \in \mathbb{C}^*\backslash\{\mathrm{e}^{\pm i\theta_j}\}_{j=0}^N, \\ 1 & \text{if } z = 0. \end{cases} \tag{4.16}$$

This mapping K fulfills the following properties:

(i) *K is analytic on* $\mathbb{C}\backslash\Gamma_{\mathbf{b}}$.

(ii) *For every* $\ell = 1, \ldots, 2N$, $K|\mathrm{Int}(\Gamma_\ell)$ *is a real and continuous function* (*regarded as a function with domain* $\mathrm{Int}(\Gamma_\ell)$) *which never vanishes*, *and*

$$K(z) = (-1)^{N-\ell} \sqrt{\frac{\rho_{N-1}\left(\frac{1}{d}\cos\theta\right)}{\Lambda_{N-1}\left(\frac{1}{d}\cos\theta\right)}}, \quad z = \mathrm{e}^{\mathrm{i}\theta} \in \mathrm{Int}(\Gamma_\ell). \tag{4.17}$$

(iii) $K(z)$ *is purely imaginary for* $z \in \partial\mathbb{D}\backslash\Gamma_{\mathbf{b}}$.

Proof. The zeros of $\Lambda_{N-1}(g(z)) \cdot (z - 1/z)$ are the numbers $\mathrm{e}^{\pm\mathrm{i}\theta_j}$ ($j = 0, 1, \ldots, N$), all of which belong to $\Gamma_{\mathbf{b}}$, and we know that Θ is analytic in $\mathbb{C}\backslash\Gamma_{\mathbf{b}}$. This implies that K is analytic on $\mathbb{C}^*\backslash\Gamma_{\mathbf{b}}$. Since Λ_{N-1} is a monic polynomial of degree $N-1$ we see that $z^{N-1}\Lambda_{N-1}(g(z)) \to (2d)^{1-N}$ as $z \to 0$. Hence, for $z \neq 0$ but near zero, and taking into account (v) in Lemma 4.2, we get

$$K(z) = \frac{d}{z^{N-1}\Lambda_{N-1}(g(z))}\frac{z^N}{\Theta(z)}\frac{\Theta^2(z) - 1}{z^2 - 1} \to \frac{d}{(2d)^{1-N}}\frac{2}{(2d)^N} = 1$$

as $z \to 0$, so that K is continuous at $z = 0$, and since it is analytic on a punctured neighborhood of $z = 0$ then K is also analytic at $z = 0$. This proves (i).

Next we prove (ii) for a fix $\ell \in \{1, \ldots, N\}$ (the proof is similar for $\ell \in \{N+1, \ldots, 2N\}$). Take $z \in \mathrm{Int}(\Gamma_\ell)$. Then $z = \mathrm{e}^{\mathrm{i}\theta}$, with $-\pi < \theta < 0$ and $(1/d)\cos\theta \in I_\ell^\circ$. Since $\pi_N((1/d)\cos\theta) \in]-1, 1[$, from (4.13) we get

$$\frac{\Theta(z) - 1/\Theta(z)}{z - 1/z} = \frac{-1}{\sin\theta}\sqrt{1 - \pi_N^2\left(\frac{1}{d}\cos\theta\right)}. \tag{4.18}$$

But, according to (3.43),

$$1 - \pi_N^2\left(\frac{1}{d}\cos\theta\right) = d^{-2}\sin^2\theta\,\rho_{N-1}\left(\frac{1}{d}\cos\theta\right)\Lambda_{N-1}\left(\frac{1}{d}\cos\theta\right).$$

In particular, this equality shows that ρ_{N-1} and Λ_{N-1} have the same sign at $(1/d)\cos\theta$, so that their product or quotient is strictly positive in I_ℓ°. Therefore

$$\sqrt{1 - \pi_N^2\left(\frac{1}{d}\cos\theta\right)} = \frac{-\sin\theta}{d}\sqrt{\rho_{N-1}\left(\frac{1}{d}\cos\theta\right)\Lambda_{N-1}\left(\frac{1}{d}\cos\theta\right)},$$

and so from (4.18) and the definition of K, and taking into account that

$$\mathrm{sgn}\left\{\Lambda_{N-1}\left(\frac{1}{d}\cos\theta\right)\right\} = \mathrm{sgn}\left\{\pi_N'\left(\frac{1}{d}\cos\theta\right)\right\} = (-1)^{N-\ell},$$

we deduce (4.17), from which we get (ii). Statement (iii) follows from (4.13) by observing that if $z = \mathrm{e}^{\mathrm{i}\theta} \in \partial\mathbb{D}\backslash\Gamma_{\mathbf{b}}$ then $\pi_N((1/d)\cos\theta)$ is real and lives off $[-1, 1]$. This completes the proof. □

4.3. Solution to Problem (**P**)

We can now state our main result. We need to introduce some appropriate monic $2nN$—associated polynomials, namely polynomials $\widetilde{\Phi}_j^{(2nN)}(z)$ and $\widetilde{\Omega}_j^{(2nN)}(z)$ defined recursively by

$$\begin{aligned}
&\widetilde{\Phi}_0^{(2nN)}(z) \equiv 1, \quad \widetilde{\Phi}_j^{(2nN)}(z) = z\widetilde{\Phi}_{j-1}^{(2nN)}(z) + \widetilde{\Phi}_{2nN+j}(0)\widetilde{\Phi}_{j-1}^{(2nN)*}(z),\\
&\widetilde{\Omega}_0^{(2nN)}(z) \equiv 1, \quad \widetilde{\Omega}_j^{(2nN)}(z) = z\widetilde{\Omega}_{j-1}^{(2nN)}(z) - \widetilde{\Phi}_{2nN+j}(0)\widetilde{\Omega}_{j-1}^{(2nN)*}(z)
\end{aligned} \tag{4.19}$$

for all $j = 1, 2, \ldots, 2N-1$ and $n = 0, 1, 2, \ldots$. General m—associated polynomials were introduced and studied by Peherstorfer in [25]. Notice that $\widetilde{\Phi}_{2nN+j}(0) = b_j$ is independent of n for $j = 1, 2, \ldots, N-1$, so that the above recurrence relations give

$$\widetilde{\Phi}_j^{(2nN)}(z) = \widetilde{\Phi}_j(z), \quad \widetilde{\Omega}_j^{(2nN)}(z) = \widetilde{\Omega}_j(z), \quad j = 1, 2, \ldots, N-1.$$

Notice also that for each $j = 1, 2, \ldots, 2N-1$ the polynomials $\widetilde{\Phi}_j^{(2nN)}(z)$ and $\widetilde{\Omega}_j^{(2nN)}(z)$ can be expressed in terms of determinants of order j, involving only z and the corresponding Verblunsky coefficients.

Theorem 4.5. *Let $\{\Phi_n\}_{n\geqslant 0}$ be a monic OPUC with respect to some measure $\mathrm{d}\mu$. Assume that $\mathrm{d}\mu$ is symmetric, so that* (4.1)–(4.2) *hold. Let $\mathbf{b} = (b_1, \ldots, b_{N-1}) \in (-1, 1)^{N-1}$ and let $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ be the monic OPUC whose Verblunsky coefficients are defined as in* (1.3). *Let Θ, K, and ϑ_N be the mappings introduced in Lemmas* 4.2 *and* 4.4. *Finally, set*

$$E_{\mathbf{b}}^{\xi,\eta} := \left\{\theta \in [0, 2\pi] : \frac{1}{d}\cos\theta \in \pi_N^{-1}([\xi, \eta])\right\} \subset E_{\mathbf{b}},$$

$\xi := \cos\theta_M$, $\eta := \cos\theta_m$. *Then the following statements hold*:

(i) *The polynomial sequence $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ is explicitly given by*

$$\widetilde{\Phi}_{2nN+j}(z) = \left(\frac{(2dz)^N}{2\Theta(z)}\right)^n \{G_j^+(z; n)\Phi_{2n}(\Theta(z)) + G_j^-(z; n)\Phi_{2n}^*(\Theta(z))\} \tag{4.20}$$

for all $j = 0, 1, \ldots, 2N-1$ and $n = 0, 1, 2, \ldots$, where

$$G_j^{\pm}(z; n) := \frac{1}{2}\left\{\widetilde{\Phi}_j^{(2nN)}(z) \pm K(z)\widetilde{\Omega}_j^{(2nN)}(z)\right\}.$$

(ii) *The Carathéodory functions associated with $\{\Phi_n\}_{n\geqslant 0}$ and $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ satisfy*

$$C(z; \mathrm{d}\widetilde{\mu}) = \frac{1}{K(z)} C(\Theta(z); \mathrm{d}\mu), \quad |z| < 1. \tag{4.21}$$

(iii) *The measure $\mathrm{d}\widetilde{\mu}$ with respect to which $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ is orthogonal is*

$$\mathrm{d}\widetilde{\mu}(\theta) = \frac{1}{d}\left|\zeta_{N-1}(\theta)\right| \left|\frac{\sin\theta}{\sin\vartheta_N(\theta)}\right| \frac{\mathrm{d}\mu(\vartheta_N(\theta))}{\vartheta_N'(\theta)}, \quad \mathrm{supp}(\mathrm{d}\widetilde{\mu}) \subset E_{\mathbf{b}}^{\xi,\eta} \tag{4.22}$$

where $\zeta_{N-1}(\theta) := \Lambda_{N-1}(\frac{1}{d}\cos\theta)$. Alternatively,

$$\mathrm{d}\widetilde{\mu}(\theta) = \frac{\chi_{E_{\mathbf{b}}^{\xi,\eta}}(\theta)}{|K(\mathrm{e}^{\mathrm{i}\theta})|} \frac{\mathrm{d}\mu(\vartheta_N(\theta))}{\vartheta_N'(\theta)}, \tag{4.23}$$

where $\chi_{E_{\mathbf{b}}^{\xi,\eta}}$ denotes the characteristic function of the set $E_{\mathbf{b}}^{\xi,\eta}$.

Proof. From the Szegö relations (2.3), for every $n = 0, 1, 2, \ldots$ we find

$$\Phi_{2n}(z) = z^n\left\{\frac{1+\Phi_{2n}(0)}{2^{1-n}} P_n(x) + \frac{1-\Phi_{2n}(0)}{2^{1-n}} \frac{z - z^{-1}}{2} Q_{n-1}(x)\right\},$$

$x=(z+z^{-1})/2$ (we choose z so that $z=x-\sqrt{x^2-1}\equiv h(x)$), and a similar relation holds for the polynomials of the sequence $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$. Hence

$$\begin{aligned}\widetilde{\Phi}_{2nN}(z)&=z^{nN}\left\{\frac{1+\widetilde{\Phi}_{2nN}(0)}{2^{1-nN}}\widetilde{P}_{nN}(x)+\frac{1-\widetilde{\Phi}_{2nN}(0)}{2^{1-nN}}\frac{z-z^{-1}}{2}\widetilde{Q}_{nN-1}(x)\right\}\\&=\left(\frac{(2dz)^N}{2}\right)^n\left\{\frac{1}{2\varepsilon_n}P_n\left(\pi_N\left(\frac{x}{d}\right)\right)+\frac{1}{\delta_n}\frac{z-z^{-1}}{2d}\rho_{N-1}\left(\frac{x}{d}\right)Q_{n-1}\left(\pi_N\left(\frac{x}{d}\right)\right)\right\},\end{aligned}$$

the last equality being justified by relations (1.3), (2.4) and Theorems 3.3 and 3.4. Put $X=\pi_N(x/d)=\frac{1}{2}(Z+Z^{-1})$, where we choose Z so that $Z=X-\sqrt{X^2-1}\equiv h(X)$, which gives $Z(z)=\Theta(z)$. Notice that, by Theorem 3.4,

$$\begin{aligned}\frac{z-z^{-1}}{2d}\rho_{N-1}\left(\frac{x}{d}\right)&=\frac{2d}{z-z^{-1}}\frac{X^2-1}{\Lambda_{N-1}\left(\frac{x}{d}\right)}\\&=\frac{d}{\Lambda_{N-1}(g(z))}\frac{Z-Z^{-1}}{z-z^{-1}}\frac{Z-Z^{-1}}{2}\\&=\frac{1}{2}K(z)(Z-Z^{-1})\end{aligned}$$

(we need to assume that z is neither ± 1 nor a zero of $\Lambda_{N-1}(g(z))$, but this is not important—cf. Remark 4.6 in below). Then

$$\widetilde{\Phi}_{2nN}(z)=\left(\frac{(2dz)^N}{2}\right)^n\left\{\frac{1}{2\varepsilon_n}P_n(X)+\frac{K(z)}{2\delta_n}(Z-Z^{-1})Q_{n-1}(X)\right\}.$$

From this equality, and taking into account that, according to (2.3), the relations

$$\frac{1}{\varepsilon_n}P_n(X)=Z^{-n}\Phi_{2n}(Z)+Z^n\Phi_{2n}(Z^{-1}),$$

$$\frac{1}{\delta_n}(Z-Z^{-1})Q_{n-1}(X)=Z^{-n}\Phi_{2n}(Z)-Z^n\Phi_{2n}(Z^{-1})$$

hold, we easily get

$$\widetilde{\Phi}_{2nN}(z)=\left(\frac{(2dz)^N}{2\Theta(z)}\right)^n\left\{\frac{1+K(z)}{2}\Phi_{2n}(\Theta(z))+\frac{1-K(z)}{2}\Phi^*_{2n}(\Theta(z))\right\},\tag{4.24}$$

which proves (4.20) for $j=0$. To prove (4.20) for $j=1,2,\ldots,2N-1$ we start by applying an identity from Corollary 3.1 in [25] to write

$$\widetilde{\Phi}_{2nN+j}(z)=\tfrac{1}{2}(\widetilde{\Phi}_{2nN}(z)+\widetilde{\Phi}^*_{2nN}(z))\widetilde{\Phi}^{(2nN)}_j(z)+\tfrac{1}{2}(\widetilde{\Phi}_{2nN}(z)-\widetilde{\Phi}^*_{2nN}(z))\widetilde{\Omega}^{(2nN)}_j(z)\tag{4.25}$$

for every $j=1,2,\ldots,2N-1$. Since $g(1/z)=g(z)$ for $z\in\mathbb{C}^*$, we have $\Theta(1/z)=\Theta(z)$ and $K(1/z)=-K(z)$, hence from (4.24) we obtain

$$\widetilde{\Phi}^*_{2nN}(z)=\left(\frac{(2dz)^N}{2\Theta(z)}\right)^n\left\{\frac{1-K(z)}{2}\Phi_{2n}(\Theta(z))+\frac{1+K(z)}{2}\Phi^*_{2n}(\Theta(z))\right\}.\tag{4.26}$$

Therefore

$$\widetilde{\Phi}_{2nN}(z)+\widetilde{\Phi}^*_{2nN}(z)=\left(\frac{(2dz)^N}{2\Theta(z)}\right)^n\{\Phi_{2n}(\Theta(z))+\Phi^*_{2n}(\Theta(z))\},$$

$$\widetilde{\Phi}_{2nN}(z)-\widetilde{\Phi}^*_{2nN}(z)=\left(\frac{(2dz)^N}{2\Theta(z)}\right)^n K(z)\{\Phi_{2n}(\Theta(z))-\Phi^*_{2n}(\Theta(z))\},$$

and using these expressions in (4.25) gives (4.20) for $j=1,2,\ldots,2N-1$.

In order to prove (ii), notice first that, by Lemmas 4.2 and 4.4, if $|z|<1$ then also $|\Theta(z)|<1$ and $K(z)\neq 0$, so that the right-hand side of (4.21) is well defined. Notice also that if $|z|<1$ then $z\notin\Gamma_{\mathbf{b}}$, hence $\pi_N(\frac{x}{d})=\pi_N(g(z))\notin[-1,1]$, so that $x/d\in\mathbb{C}\backslash\pi_N^{-1}([\xi,\eta])$, since $[\xi,\eta]\subset[-1,1]$ (in fact, $[\xi,\eta]$ is the true interval of orthogonality of $\{P_n\}_{n\geqslant 0}$). Then from (2.9) and Theorem 3.8, for $|z|<1$ we can write

$$\begin{aligned}C(z;\mathrm{d}\widetilde{\mu})&=\frac{z^{-1}-z}{2}F(x;\mathrm{d}\widetilde{\sigma})=\frac{z^{-1}-z}{2d}\Lambda_{N-1}\left(\frac{x}{d}\right)F\left(\pi_N\left(\frac{x}{d}\right);\mathrm{d}\sigma\right)\\&=\frac{\Lambda_{N-1}(g(z))}{d}\frac{z-z^{-1}}{Z-Z^{-1}}\frac{Z^{-1}-Z}{2}F(X;\mathrm{d}\sigma)\\&=\frac{1}{K(z)}C(\Theta(z);\mathrm{d}\mu).\end{aligned}$$

Finally, as a consequence of Theorem 3.8 and Szegö transformation between $\mathrm{d}\widetilde{\mu}$ and $\mathrm{d}\widetilde{\sigma}$, we see that $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ is orthogonal with respect to the measure

$$\mathrm{d}\widetilde{\mu}(\theta)=|\mathrm{d}\widetilde{\sigma}(\cos\theta)|=-\mathrm{sgn}\{\sin\theta\}\frac{|\Lambda_{N-1}((1/d)\cos\theta)|}{\pi_N'((1/d)\cos\theta)}\mathrm{d}\sigma\left(\pi_N\left(\frac{1}{d}\cos\theta\right)\right)$$

$(0\leqslant\theta\leqslant 2\pi)$, with support contained in $E_{\mathbf{b}}^{\xi,\eta}$, which by (4.11) and Szegö transformation between $\mathrm{d}\mu$ and $\mathrm{d}\sigma$ leads to (4.22). The equivalent form (4.23) can be easily justified by using (4.18) together with (4.11). □

Remark 4.6. We have not defined $K(z)$ at the zeros of $\Lambda_{N-1}(g(z))(z-1/z)$, which are the points $z=\mathrm{e}^{\pm\mathrm{i}\theta_j}$ $(j=0,1,\ldots,N)$. However, it is not important how K is defined at these points, since if we rewrite (4.20) as

$$\begin{aligned}\widetilde{\Phi}_{2nN+j}(z)=\frac{1}{2}\left(\frac{(2dz)^N}{2\Theta(z)}\right)^n&\{\widetilde{\Phi}^{(2nN)}_j(z)[\Phi_{2n}(\Theta(z))+\Phi^*_{2n}(\Theta(z))]\\&+K(z)\widetilde{\Omega}^{(2nN)}_j(z)[\Phi_{2n}(\Theta(z))-\Phi^*_{2n}(\Theta(z))]\}\end{aligned}$$

then we see that $K(z)$ appears as a factor in a product where $\Phi_{2n}(\Theta(z))-\Phi^*_{2n}(\Theta(z))$ is also a factor, and this factor vanishes at all the points $z=\mathrm{e}^{\pm\mathrm{i}\theta_j}$. In fact, since $\Theta(\mathrm{e}^{\pm\mathrm{i}\theta_j})=\pi_N(z_j)=(-1)^{N-j}$ for each $j=0,1,\ldots,N$, then either $\Theta(\mathrm{e}^{\pm\mathrm{i}\theta_j})=1$ or $\Theta(\mathrm{e}^{\pm\mathrm{i}\theta_j})=-1$, and it is clear by definition of reversed polynomial that $\Phi_{2n}(1)-\Phi^*_{2n}(1)=0=\Phi_{2n}(-1)-\Phi^*_{2n}(-1)$.

Remark 4.7. From (iv) in Lemma 4.2, the mapping $\theta\in E_\ell\mapsto\vartheta_N(\theta)\in[0,2\pi]$ is bijective and continuous *a.e.* for all $\ell=1,2,\ldots,2N$, so the precise meaning of (4.22) is

$$\int_{E_{\mathbf{b}}^{\xi,\eta}}f(\theta)\,\mathrm{d}\widetilde{\mu}(\theta)=\frac{1}{d}\sum_{\ell=1}^{2N}\int_{E_\ell}f(\theta)\left|\zeta_{N-1}(\theta)\right|\left|\frac{\sin\theta}{\sin\vartheta_N(\theta)}\right|\frac{\mathrm{d}\mu(\vartheta_N(\theta))}{\vartheta_N'(\theta)},$$

for every $f\in\mathscr{C}[0,2\pi]$. Similarly for the meaning of (4.23).

Remark 4.8. $\mathrm{d}\widetilde{\mu}$ induces a measure on $\partial\mathbb{D}$ which, in general, will be supported on the set $\Gamma_{\mathbf{b}}^{\xi,\eta} := \{\mathrm{e}^{\mathrm{i}\theta} : \theta \in E_{\mathbf{b}}^{\xi,\eta}\} \subset \Gamma_{\mathbf{b}}$, which is the union of $2N$ arcs on the unit circle, pairwise symmetric with respect to the real axis. In particular, if $b_1 = \cdots = b_{N-1} = 0$ and $[\xi, \eta] = [-1, 1]$ then $\Gamma_{\mathbf{b}}^{\xi,\eta} \equiv \Gamma_{\mathbf{0}} \equiv \partial\mathbb{D}$ (cf. Corollary 4.10 in bellow).

Remark 4.9. For the special case $\alpha_{2j} =$ constant for all $j \in \mathbb{N}_0$ Theorem 4.5 is due to Peherstorfer and Steinbauer [30].

Corollary 4.10 (*Badkov [3], Ismail and Li [19], Marcellan and Sansigre [22]*). *Let $b_1 = \cdots = b_{N-1} = 0$ in Theorem* 4.5. *Then*

$$\mathrm{d}\widetilde{\mu}(\theta) = \frac{1}{N}\,\mathrm{d}\mu(N\theta), \quad 0 \leqslant \theta < 2\pi, \tag{4.27}$$

$$C(z; \mathrm{d}\widetilde{\mu}) = C(z^N; \mathrm{d}\mu), \quad |z| < 1, \tag{4.28}$$

$$\widetilde{\Phi}_{nN+j}(z) = z^j \Phi_n(z^N), \quad j = 0, 1, \ldots, N-1; \;\; n = 0, 1, 2, \ldots. \tag{4.29}$$

Proof. Assume $b_1 = \cdots = b_{N-1} = 0$. In this case the arcs Γ_ℓ are explicitly given by

$$\Gamma_\ell = \left\{\mathrm{e}^{\mathrm{i}\theta} : \frac{\ell - 1 - N}{N}\pi \leqslant \theta \leqslant \frac{\ell - N}{N}\pi\right\}, \quad \ell = 1, 2, \ldots, 2N. \tag{4.30}$$

In fact, by Remark 3.5 we have $\Lambda_{N-1}(x) = \rho_{N-1}(x) = dU_{N-1}(dx)$, hence by previous considerations, the extreme points of the arcs Γ_ℓ are the points $\mathrm{e}^{\pm\mathrm{i}\theta_j}$ such that $-\pi \equiv \theta_0 < \theta_1 < \cdots < \theta_{N-1} < \theta_N \equiv 0$ and $(1/d)\cos\theta_j = z_j$, where $z_0 = -1/d$, $z_N = 1/d$ and $z_1 < \cdots < z_{N-1}$ are the zeros of $U_{N-1}(dz)$. As a consequence, $z_j = -(1/d)\cos(j/N)\pi$, hence $\theta_j = (j - N)/N\pi$ for all $j = 0, 1, \ldots, N$, and (4.30) follows from $\Gamma_\ell = \{\mathrm{e}^{\mathrm{i}\theta} : \theta_{\ell-1} \leqslant \theta \leqslant \theta_\ell\}$ for $\ell = 1, 2, \ldots, N$ and since Γ_{N+j} and Γ_{N+1-j} are symmetric with respect to the real axis for each $j = 1, 2, \ldots, N$. Notice that (4.30) implies that $\Gamma_{\mathbf{b}}$ is the whole unit circle: $\Gamma_{\mathbf{b}} = \bigcup_{\ell=1}^{2N} \Gamma_\ell = \partial\mathbb{D}$.

Now we compute the mappings Θ and K. According to Remark 3.5 we can write

$$\pi_N(g(z)) = T_N\left(\frac{z + z^{-1}}{2}\right) = \frac{z^N + z^{-N}}{2},$$

$$\Lambda_{N-1}(g(z)) = \rho_{N-1}(g(z)) = dU_{N-1}\left(\frac{z + z^{-1}}{2}\right) = d\frac{z^N - z^{-N}}{z - z^{-1}}.$$

Hence, on one hand we have

$$\Theta(z) = \begin{cases} z^N & \text{if } |z| < 1, \\ \cos(N\theta) - \mathrm{i}|\sin(N\theta)| & \text{if } z = \mathrm{e}^{\mathrm{i}\theta} \quad (-\pi \leqslant \theta < \pi), \\ z^{-N} & \text{if } |z| > 1, \end{cases}$$

where the last equality is justified by the choice of the branch of the square root. According to (4.30) we can also write

$$\Theta(z) = \begin{cases} z^N & \text{if } |z| < 1, \\ z^{(-1)^{N-\ell}N} & \text{if } z = \mathrm{e}^{\mathrm{i}\theta},\ \dfrac{\ell - 1 - N}{N}\pi \leqslant \theta \leqslant \dfrac{\ell - N}{N}\pi \quad (\ell = 1, \ldots, 2N), \\ z^{-N} & \text{if } |z| > 1. \end{cases} \tag{4.31}$$

On the other hand, for K, using the definition as well as (4.17), we find

$$K(z) = \begin{cases} 1 & \text{if } |z| < 1, \\ (-1)^{N-\ell} & \text{if } z = \mathrm{e}^{\mathrm{i}\theta},\ \dfrac{\ell - 1 - N}{N}\pi < \theta < \dfrac{\ell - N}{N}\pi \quad (\ell = 1, \ldots, 2N), \\ -1 & \text{if } |z| > 1. \end{cases}$$

It follows immediately from (4.21) that (4.28) holds. Also, using again Remark 3.5 with $\cos\theta = dx$ we find

$$\zeta_{N-1}(\theta) = dU_{N-1}(\cos\theta) = d\sin(N\theta)/\sin\theta, \quad \vartheta_N(\theta) = N\theta,$$

so that (4.22) gives (4.27). Now, since $b_1 = \cdots = b_{N-1} = 0$, we have

$$\widetilde{\Phi}_j^{(2nN)}(z) = \begin{cases} z^j, & j = 0, 1, \ldots, N-1, \\ z^{j-N}(z^N + \Phi_{2n+1}(0)), & j = N, N+1, \ldots, 2N-1, \end{cases}$$

$$\widetilde{\Omega}_j^{(2nN)}(z) = \begin{cases} z^j, & j = 0, 1, \ldots, N-1, \\ z^{j-N}(z^N - \Phi_{2n+1}(0)), & j = N, N+1, \ldots, 2N-1. \end{cases}$$

Hence, in order to prove (4.29), notice first that if $|z| < 1$ then

$$G_j^+(z; n) = z^j, \quad j = 0, 1, \ldots, 2N-1,$$

$$G_j^-(z; n) = \begin{cases} 0, & j = 0, 1, \ldots, N-1, \\ z^{j-N}\Phi_{2n+1}(0), & j = N, N+1, \ldots, 2N-1; \end{cases}$$

and if $|z| > 1$ then

$$G_j^+(z; n) = \begin{cases} 0, & j = 0, 1, \ldots, N-1, \\ z^{j-N}\Phi_{2n+1}(0), & j = N, N+1, \ldots, 2N-1, \end{cases}$$

$$G_j^-(z; n) = z^j, \quad j = 0, 1, \ldots, 2N-1.$$

(For $|z| = 1$ the computations are similar.) In any case, substituting these expressions in (4.20)—take into account that $(2dz)^N/2\Theta(z)$ equals 1 if $|z| < 1$ and it equals z^{2N} if $|z| > 1$, and use the recurrence relation for $\{\Phi_n\}_{n\geqslant 0}$—gives

$$\widetilde{\Phi}_{2nN+j}(z) = \begin{cases} z^j\Phi_{2n}(z^N), & j = 0, 1, \ldots, N-1, \\ z^{j-N}\Phi_{2n+1}(z^N), & j = N, N+1, \ldots, 2N-1 \end{cases}$$

for all $n = 0, 1, 2, \ldots$, which is equivalent to (4.29). $\square$

Remark 4.11. It is a curious fact that the mapping $z \mapsto \Theta(z)$ corresponding to the case $\mathbf{b} = \mathbf{0}$ is not given by $\Theta(z) = z^N$ in the whole complex plane (as could be expected), but in fact it is given by (4.31).

4.4. Asymptotic behavior of the perturbed polynomials

Recall that the measure $\mathrm{d}\mu$ belongs to Szegö's class if $\{\Phi_n(0)\}_{n\geqslant 0} \in \ell^2$, the Hilbert sequence space of all square-summable sequences. This is equivalent to the integrability of the logarithm of the absolutely continuous part in the Lebesgue decomposition of $\mathrm{d}\mu$. In this case, the asymptotic behavior of the OPs $\{\Phi_n\}_{n\geqslant 0}$ can be obtained by using the so-called Szegö function,

$$D(z; \mathrm{d}\mu) := \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z}\log\mu'(\theta)\,\mathrm{d}\theta\right\}, \quad |z| < 1.$$

In fact, the asymptotic results

$$\lim_{n\to+\infty}\varphi_n(z) = 0, \quad \lim_{n\to+\infty}\varphi_n^*(z) = \frac{1}{D(z; \mathrm{d}\mu)}$$

hold uniformly in compact subsets of the unit disk $\mathbb{D}$ (cf. [36,15,16,34]), φ_n being the orthonormal polynomial corresponding to the monic polynomial Φ_n, so

$$\varphi_n(z) = k_n\Phi_n(z), \quad k_n := \prod_{j=1}^n \frac{1}{\sqrt{1 - |\Phi_j(0)|^2}} \tag{4.32}$$

$(n = 0, 1, 2, \ldots)$ and φ_n^* is the reversed polynomial of φ_n (see Simon [32, p. 1]). If $\mathrm{d}\mu$ belongs to the Szegö class then it is clear that the same does not hold for the measure $\mathrm{d}\tilde{\mu}$ in Problem (**P**), up to $\mathbf{b} \equiv (0, \ldots, 0)$, since otherwise $\{\widetilde{\Phi}_n(0)\}_{n\geqslant 0} \notin \ell^2$. However, the asymptotic behavior for the orthonormal polynomials can be easily obtained as the next proposition shows.

Theorem 4.12. *Let the assumptions in Theorem* 4.5 *hold, and assume that* $\mathrm{d}\mu$ *belongs to the Szegö class and let* $\{\widetilde{\varphi}_n\}_{n\geqslant 0}$ *be the corresponding sequence of orthonormal polynomials. Under these conditions,*

(i) *for each* $j = 0, 1, \ldots, 2N-1$ *the limit* $G_j(z) := \lim_{n\to+\infty} G_j^-(z; n)$ *exists for every* $z \in \mathbb{C}\backslash\Gamma_{\mathbf{b}}$, *and*

$$G_j(z) = \begin{cases} G_j^-(z;0) & \text{if } j = 0, 1, \ldots, N-1, \\ zG_{N-1}^-(z;0) & \text{if } j = N, \\ zG_{j-1}(z) - b_{2N-j}G_{j-1}^*(z) & \text{if } j = N+1, \ldots, 2N-1 \end{cases}$$

(*with the usual definition* $G_{j-1}^*(z) := z^{j-1}G_{j-1}(1/z)$, *even if* $G_{j-1}(z)$ *is not a polynomial*).

(ii) *Strong asymptotics*

$$\lim_{n\to+\infty} \left(\frac{\Theta(z)}{z^N}\right)^n \widetilde{\varphi}_{2nN+j}(z) = \frac{v_j G_j(z)}{D(\Theta(z); \mathrm{d}\mu)}, \quad j = 0, 1, \ldots, 2N-1$$

hold uniformly in compact subsets of $\mathbb{C}\backslash\Gamma_{\mathbf{b}}$, *where* v_j *is a constant defined by*

$$v_j := \begin{cases} \displaystyle\prod_{s=1}^{j} \frac{1}{\sqrt{1-b_s^2}} & \text{if } j = 0, 1, \ldots, N-1 \\ \displaystyle\frac{\sqrt{2}}{(2d)^{N/2}} \prod_{s=1}^{j-N} \frac{1}{\sqrt{1-b_{N-s}^2}} & \text{if } j = N, N+1, \ldots, 2N-1. \end{cases}$$

Proof. As in (4.32), we have $\widetilde{\varphi}_n(z) = \widetilde{k}_n\widetilde{\Phi}_n(z)$ with $\widetilde{k}_n := \prod_{j=1}^n 1/\sqrt{1-|\widetilde{\Phi}_j(0)|^2}$. According to (1.3), it is straightforward to verify that

$$\widetilde{k}_{2nN+j} = \begin{cases} \left(2/(2d)^N\right)^n v_j k_{2n}, & j = 0, 1, \ldots, N-1 \\ (2/(2d)^N)^n v_j k_{2n+1}, & j = N, N+1, \ldots, 2N-1. \end{cases}$$

Hence, taking into account the relation $k_{2n+1} = k_{2n}/\sqrt{1-\Phi_{2n+1}^2(0)}$ and the fact $\Phi_n(0) \to 0$ as $n \to +\infty$ (since $\mathrm{d}\mu$ belongs to Szegö class), we have

$$\lim_{n\to+\infty} \frac{\widetilde{k}_{2nN+j}}{k_{2n}} \left(\frac{(2d)^N}{2}\right)^n = v_j, \quad j = 0, 1, \ldots, 2N-1.$$

Now, by (4.20) in Theorem 4.5,

$$\left(\frac{\Theta(z)}{z^N}\right)^n \widetilde{\varphi}_{2nN+j}(z) = \frac{\widetilde{k}_{2nN+j}}{k_{2n}} \left(\frac{(2d)^N}{2}\right)^n \{G_j^+(z;n)\varphi_{2n}(\Theta(z)) + G_j^-(z;n)\varphi_{2n}^*(\Theta(z))\}.$$

But, by Lemma 4.2, if $z \in \mathbb{C}\backslash\Gamma_{\mathbf{b}}$ then $|\Theta(z)| < 1$ and compact subsets of $\mathbb{C}\backslash\Gamma_{\mathbf{b}}$ are mapped by Θ in compact subsets of the unit circle $|z| < 1$, and so, since $\mathrm{d}\mu$ belongs to Szegö class,

$$\lim_{n\to+\infty} \varphi_{2n}(\Theta(z)) = 0, \quad \lim_{n\to+\infty} \varphi_{2n}^*(\Theta(z)) = 1/D(\Theta(z); \mathrm{d}\mu)$$

uniformly in compact subsets of $\mathbb{C}\backslash\Gamma_{\mathbf{b}}$. Since $G_j^{\pm}(z;n)$ are independent of n for all $j = 0, 1, \ldots, N-1$, we get (i) and (ii) for these values of j. For $j = N, N+1, \ldots, 2N-1$ we notice that both $\lim_{n\to+\infty} G_j^{\pm}(z;n)$ exist for every fixed $z \in \mathbb{C}\backslash\Gamma_{\mathbf{b}}$, which can be easily seen by using the determinantal expressions for $\widetilde{\Phi}_j^{(2nN)}(z)$ and $\widetilde{\Omega}_j^{(2nN)}(z)$ (as determinants of order j, in terms of z and the corresponding Verblunsky coefficients), and taking into account that $\Phi_n(0) \to 0$ as $n \to +\infty$. Further, (4.19) gives the recurrence

$$G_j^-(z;n) = zG_{j-1}^-(z;n) + \widetilde{\Phi}_{2nN+j}(0)[G_{j-1}^-(z;n)]^*$$

(for $j=1,\ldots,2N-1$). Therefore, for $j=N$ we get $G_N^-(z;n)=zG_{N-1}^-(z;0)+\Phi_{2n+1}(0)[G_{N-1}^-(z;0)]^*$, so $G_N^-(z;n) \to zG_{N-1}^-(z;0)$ as $n \to +\infty$; and for $j=N+1,\ldots,2N-1$, since $G_j^-(z;n)=zG_{j-1}^-(z;n)-b_{2N-j}[G_{j-1}^-(z;n)]^*$, by taking the limit as $n \to +\infty$ we obtain a recurrence relation for $G_j(z)$. This completes the proof. □

Remark 4.13. By the assumptions of Theorem 4.12 the Verblunsky coefficients for the sequence $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ are asymptotically periodic. Therefore, under the assumption $\sum_{n\geqslant 0}|a_n| < \infty$ (which needs not to hold if we only assume Szegö's condition), the limit statements given for such sequence by Peherstorfer and Steinbauer in [31, Section 3] can be applied in order to get the strong asymptotics for the perturbed sequence $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$.

Remark 4.14. From the asymptotics of the perturbed OPUC $\{\widetilde{\Phi}_n\}_{n\geqslant 0}$ we can now find the strong asymptotics of the perturbed OPRL $\{\widetilde{P}_n\}_{n\geqslant 0}$. Further, information concerning the spectral properties of the Jacobi operator with entries $\{\widetilde{a}_n\}_{n\geqslant 1} \subset \mathbb{R}^+$ and $\{\widetilde{b}_n\}_{n\geqslant 0} \subset \mathbb{R}$, where $\widetilde{b}_n=\widetilde{\beta}_n$ and $\widetilde{a}_n^2=\widetilde{\gamma}_n$, $\{\widetilde{\beta}_n,\widetilde{\gamma}_{n+1}\}_{n\geqslant 0}$ being the sequences introduced in (3.35), can be obtained (in terms of the spectral properties of the Jacobi operator corresponding to the OPRL $\{P_n\}_{n\geqslant 0}$).

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